Feedback Regularization and Control of Nonlinear Differential-Algebraic-Equation Systems

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The feedback control of nonlinear high-index differential-algebraic-equation systems for which the underlying algebraic constraints among the system variables involve the manipulated inputs is addressed in this work. A state-space realization of such systems cannot be derived independently of the controller design. In view of this fact, a two-step methodology is proposed for the control of such systems. The first step involves the derivation of a dynamic state-feedback compensator such that in the resulting system, the underlying constraints are independent of the new inputs. In the second step, a state-space realization of the feedback modified system is derived and used as the basis for a state-feedback controller synthesis. Application of the developed control methodology is demonstrated on an interconnection of a two-phase exothermic reactor and a condenser.

Introduction

Systems of coupled nonlinear differential and algebraic equations (DAEs) arise naturally in a variety of chemical engineering processes, where the dynamic balances of mass and energy constitute the differential equations, while the algebraic equations include thermodynamic equilibrium relations, empirical correlations, and pseudo-steady-state conditions (see, e.g., Byrne and Ponzi, 1988; Gani and Cameron, 1992; Pantelides, 1988; Pantelides et al., 1988). Extensive research over the last two decades on numerical analysis and simulation aspects (see Brenan et al., 1989, and references therein) has established that DAE systems differ fundamentally different from systems of pure ordinary differential equations (ODEs). An important concept that provides a measure of these differences is that of the (differential) index (Brenan et al., 1989). The index of a DAE system is defined as the minimum number of differentiations required to convert the system into an equivalent ODE system. While DAE systems with index-one are similar to ODE systems, high-index DAE systems differ significantly (Petzold, 1982). Typical examples of chemical processes that are modeled by high-index DAE systems include multiphase reactors and absorption/distillation columns with phase equilibrium, process networks with negligible pressure drop, and reactors with slow and fast reactions where the fast reversible ones are at equilibrium and the fast irreversible ones yield the pseudo-steady-state conditions of complete conversion for the reactants.

High-index DAE systems are characterized by the presence of additional underlying algebraic constraints among the system variables, which lead to nontrivial problems in consistent initialization (Leimkuhler et al., 1991; Pantelides, 1988) and numerical simulation (Brenan et al., 1989). In view of these difficulties, the numerical simulation of general high-index DAE systems relies on various index-reduction techniques to obtain equivalent systems that can be solved efficiently through ODE integration methods (e.g., Bachmann et al., 1990; Chung and Westerberg, 1990; Gear and Petzold, 1984; Gear, 1988). The results on numerical analysis and simulation of DAE systems have also been used to address the control of ODE systems, where the prescribed closed-loop responses are viewed as the algebraic constraint equations to obtain a DAE formulation of the control problem (McLellan, 1994; Pantelides, 1988).

The majority of research on control of DAE systems has focused on linear systems, often referring to them as singular, semistate, or descriptor systems (see, e.g., Campbell, 1980, 1982), with particular emphasis on optimal control (Bender and Laub, 1987; Jonckheere, 1988) and elimination of impulsive behavior through feedback (Cobb, 1981). Research on control of nonlinear DAE systems has been rather limited until recently, with few results on optimal control (e.g., Cuthrell and Biegler, 1989; Pantelides et al., 1992), and state feedback stabilization and tracking of a class of DAE systems

(Krishnan and McClamroch, 1993; McClamroch, 1990) that arise mainly in mechanical systems. A methodological framework for the feedback control of a more general class of nonlinear, high-index, semiexplicit DAE systems was introduced in Kumar and Daoutidis (1995a), and was further generalized in Kumar and Daoutidis (1995b) to address the control of DAE systems with disturbances. The developed methodology involved (1) the derivation of a state-space realization of the constrained system, and (2) the formulation and solution of a feedback controller synthesis problem on the basis of the resulting state-space realization. Owing to its sequential nature, the two-step approach was naturally limited to DAE systems for which the underlying algebraic constraints were independent of the manipulated inputs, and thus a state-space realization could be derived in step (1) independently of the controller design in step (2). However, depending on the choice of the manipulated inputs, the underlying algebraic constraints in a high-index DAE system may involve these inputs as well. This work addresses the feedback control of nonlinear high-index DAE systems, for which the underlying algebraic constraints involve the manipulated inputs. For such systems, the presence of the manipulated inputs in the algebraic constraints implies that the state-space, and hence the state-space realization *depend* on the feedback control law. In the light of the aforementioned fundamental difference from the class of systems considered in Kumar and Daoutidis (1995a), a novel control methodology is developed, where the key step involves the derivation of a dynamic state-feedback compensator to modify the DAE system such that the underlying algebraic constraints in the resulting system are independent of the new inputs. Thereafter, a state-space realization of the feedback modified system is derived and subsequently used as the basis for a state-feedback controller synthesis. The developed control methodology is applied to an interconnection of a vapor–liquid reactor and a condenser/separator, with liquid recycle.

Preliminaries and Methodological Framework

We consider multi-input multioutput (MIMO) nonlinear DAE systems with a description of the form:

$$\begin{aligned}\dot{x} &= f(x) + b(x)z + g(x)u \\ 0 &= k(x) + l(x)z + c(x)u \\ y_i &= h_i(x), \qquad i = 1, \dots, m,\end{aligned} \tag{1}$$

where $x \in \mathfrak{X} \subset \mathbb{R}^n$ is the vector of differential variables (those for which we have explicit differential equations), $z \in \mathcal{Z} \subset \mathbb{R}^p$ is the vector of algebraic variables ($\mathfrak{X}$, $\mathcal{Z}$ are open sets of dimensions n, p, respectively), $u \in \mathbb{R}^m$ is the vector of manipulated inputs, and y_i, $i = 1, \dots, m$, are the outputs to be controlled. Furthermore, $f(x)$, $k(x)$ are analytic vector fields of dimensions n, p, respectively, $h_i(x)$ are analytic scalar functions, while $b(x)$, $g(x)$, $l(x)$, and $c(x)$ are analytic matrices of appropriate dimensions. Note that the manipulated inputs u and the algebraic variables z appear in an affine and separable fashion, which covers a wide majority of practical applications and facilitates an explicit analysis and controller synthesis. More general systems that are nonlinear in u and/or z can also be easily recast in the form of Eq. 1 through an appropriate dynamic extension (Kumar and Daoutidis, 1995c) (see also the application example).

It is assumed that the DAE system in Eq. 1 is solvable, that is, for a consistent set of initial conditions ($x(0)$, $z(0)$) and smooth inputs $u(t)$, there exists locally, a unique smooth solution ($x(t)$, $z(t)$). The index ν_d for such systems is the minimum number of times the algebraic equations have to be differentiated to obtain differential equations for the algebraic variables z (Brenan et al., 1989). Clearly, DAE systems of Eq. 1 have an index-one, if the matrix $l(x)$ is nonsingular. For such systems, the algebraic equations can be solved for the algebraic variables z in terms of x and u, to obtain an equivalent ODE representation.

In this work, we focus on high-index DAE systems, for which the matrix $l(x)$ is singular. Owing to the singular nature of the algebraic equations, and consequently, the presence of underlying constraints in x, the representation in Eq. 1 does not constitute a standard state-space description. Thus, key control-theoretic concepts and the formulation and solution of a meaningful control problem on the basis of the representation in Eq. 1 are obscure. These issues can be addressed efficiently on the basis of a state-space realization of the constrained system. The derivation of such a state-space realization entails:

(1) the specification of the state space where the differential variables x are constrained to evolve, and

(2) the solution of algebraic variables z as a function of x and u to obtain a set of explicit differential equations for x on the constrained state space.

Note that the state-space realization of the constrained system must be independent of the controller, if it is to form the basis for a feedback controller synthesis problem. In particular, the state space where the constrained system evolves must be invariant under the feedback control law for u. Motivated by this, we introduce the following notion of regularity.

Definition. A DAE system of the form of Eq. 1 will be said to be regular, if

(1) it is solvable, with a finite index ν_d, and

(2) the state space where the differential variables x are constrained to evolve is invariant under any control law.

The problem of deriving state-space realizations of regular DAE systems was addressed in Kumar and Daoutidis (1995a) within the perspective of nonlinear inversion of ODE systems. More specifically, an algorithmic procedure based on Hirschorn's inversion algorithm (Hirschorn, 1979) was developed for this purpose. It involved, in each iteration, the identification of underlying constraints in x imposed by the singular algebraic equations. These constraints, which are independent of u for a regular system, were differentiated to obtain the new set of algebraic equations for the succeeding iteration. The algorithmic procedure converged with a final set of algebraic equations that are solvable in z, thus enabling the derivation of state-space realizations that formed the basis for the feedback controller synthesis.

In this work, we focus on DAE systems of Eq. 1 for which the underlying constraints in x involve the manipulated inputs u. Such systems are clearly nonregular, since the state space of the constrained system, characterized by the underlying constraints in x, depends on the control law for u. This fact indicates that unlike the case with regular systems, for nonregular systems the derivation of state-space realizations and the synthesis of feedback controllers are inherently coupled, and thus they must be addressed simultaneously. Motivated by this, a general control methodology is developed for nonregular DAE systems of the form of Eq. 1, which involves the following two steps:

(1) In the first (and the key) step, a state-feedback compensator is designed that results in a modified DAE system, which is regular. To this end, initially, an algorithmic procedure is developed to obtain a new DAE system, equivalent to the system in Eq. 1, with the algebraic equations explicitly including the constraints in x that involve the manipulated inputs u, thereby isolating the cause of nonregularity. The desired feedback regularizing compensator is then derived for this equivalent DAE system.

(2) In the second step, state-space realizations of the feedback-regularized DAE system are derived and used as the basis for the synthesis of a state-feedback controller that induces a well-characterized input/output behavior in the closed-loop system with stability.

The requisite algorithmic procedure is developed in the next section, followed by the feedback regularization and the controller synthesis.

Algorithmic Procedure for Derivation of Equivalent DAE System with Explicit Constraints in x Involving u

The algorithmic procedure involves, in each iteration, (a) elementary row operations on the algebraic equations to identify the underlying constraints in x, of which a *minimal* number involve the inputs u, and (b) selective differentiation of only those constraints that do not involve u, to obtain the algebraic equations for the next iteration. The procedure thus avoids introducing any derivatives of the inputs u, and finally yields an equivalent DAE system for which a feedback regularizing compensator can be designed.

Iteration 1

Consider algebraic equations of the DAE system in Eq. 1:

$$0 = k(x) + l(x)z + c(x)u, \tag{2}$$

where rank $l(x) = p_1 < p$ and rank $[l(x)\ c(x)] = m_1 \le p$ ($m_1 \ge p_1$). Then, there exists a nonsingular analytic $p \times p$ matrix $E^1(x)$ such that

$$E^1(x)[l(x)\ c(x)] = \begin{bmatrix} \bar{l}^1(x) & \bar{c}^1(x) \\ 0 & \hat{c}^1(x) \\ 0 & 0 \end{bmatrix} \tag{3}$$

where $\bar{c}^1(x)$, $\hat{c}^1(x)$ are matrices of dimensions $p_1 \times m$, $(m_1 - p_1) \times m$, respectively, and the $p_1 \times p$, $m_1 \times (p+m)$ matrices

$$\bar{l}^1(x), \qquad \begin{bmatrix} \bar{l}^1(x) & \bar{c}^1(x) \\ 0 & \hat{c}^1(x) \end{bmatrix}$$

have full row rank.

Step 1. Premultiply the algebraic equations (Eq. 2) by the matrix $E^1(x)$ to obtain:

$$0 = \begin{bmatrix} \bar{k}^1(x) \\ \hat{k}^1(x) \\ k^1(x) \end{bmatrix} + \begin{bmatrix} \bar{l}^1(x) \\ 0 \\ 0 \end{bmatrix} z + \begin{bmatrix} \bar{c}^1(x) \\ \hat{c}^1(x) \\ 0 \end{bmatrix} u \tag{4}$$

where $\bar{k}^1(x)$, $\hat{k}^1(x)$, $\boldsymbol{k}^1(x)$ are analytic vector fields of dimensions p_1, $(m_1 - p_1)$ and $(p - m_1)$, respectively. The last $p - p_1$ equations in Eq. 4 denote underlying constraints in x; a minimal number of these constraints (the first $m_1 - p_1$) involve the inputs u in an irreducible fashion, that is, $\hat{c}^1(x)$ has full row rank.

Step 2. Differentiate the last $p - m_1$ constraints that do not involve u to obtain the following new set of algebraic equations:

$$0 = \begin{bmatrix} \bar{k}^1(x) \\ \hat{k}^1(x) \\ \tilde{k}^2(x) \end{bmatrix} + \begin{bmatrix} \bar{l}^1(x) \\ 0 \\ \tilde{l}^2(x) \end{bmatrix} z + \begin{bmatrix} \bar{c}^1(x) \\ \hat{c}^1(x) \\ \tilde{c}^2(x) \end{bmatrix} u. \tag{5}$$

In the preceding equation, $\tilde{k}^2(x) = [L_f \boldsymbol{k}_1^1(x) \ \cdots \ L_f \boldsymbol{k}_{p-m_1}^1(x)]^T$, where $\boldsymbol{k}_i^1(x)$ denotes the ith component of the vector field $\boldsymbol{k}^1(x)$, and $\tilde{l}^2(x)$, $\tilde{c}^2(x)$ are matrices of dimensions $(p - m_1) \times p$, $(p - m_1) \times m$, respectively, defined as

$$\tilde{l}^2(x) = \begin{bmatrix} L_{b_1}\boldsymbol{k}_1^1(x) & \cdots & L_{b_p}\boldsymbol{k}_1^1(x) \\ \vdots & & \vdots \\ L_{b_1}\boldsymbol{k}_{p-m_1}^1(x) & \cdots & L_{b_p}\boldsymbol{k}_{p-m_1}^1(x) \end{bmatrix},$$

$$\tilde{c}^2(x) = \begin{bmatrix} L_{g_1}\boldsymbol{k}_1^1(x) & \cdots & L_{g_m}\boldsymbol{k}_1^1(x) \\ \vdots & & \vdots \\ L_{g_1}\boldsymbol{k}_{p-m_1}^1(x) & \cdots & L_{g_m}\boldsymbol{k}_{p-m_1}^1(x) \end{bmatrix},$$

where $b_j(x)$, $g_j(x)$ denote the jth columns of corresponding matrices, and $L_f \boldsymbol{k}_i^1(x)$, $L_{b_j}\boldsymbol{k}_i^1(x)$, $L_{g_j}\boldsymbol{k}_i^1(x)$ denote standard Lie derivatives (for a definition, see the Appendix).

Step 3. Evaluate

$$\text{rank}\begin{bmatrix} \bar{l}^1(x) \\ \tilde{l}^2(x) \end{bmatrix} = p_2, \ \text{rank}\begin{bmatrix} \bar{l}^1(x) & \bar{c}^1(x) \\ 0 & \hat{c}^1(x) \\ \tilde{l}^2(x) & \tilde{c}^2(x) \end{bmatrix} = m_2, \ m_2 \ge p_2.$$

If $m_2 = p$, then stop, else proceed to the next iteration. For the case when $m_1 = p$, the procedure converges after Step 1, and all the $p-p_1$ constraints in x, that is, $0=\hat{k}^1(x)+\hat{c}^1(x)u$, involve u in an irreducible fashion.

Iteration q ($q \geq 2$)

Consider the algebraic equations from iteration $q-1$:

$$0=\begin{bmatrix}\bar{k}^{q-1}(x)\\ \hat{k}^{q-1}(x)\\ \tilde{k}^{q}(x)\end{bmatrix}+\begin{bmatrix}\bar{l}^{q-1}(x)\\ 0\\ \tilde{l}^{q}(x)\end{bmatrix}z+\begin{bmatrix}\bar{c}^{q-1}(x)\\ \hat{c}^{q-1}(x)\\ \tilde{c}^{q}(x)\end{bmatrix}u, \tag{6}$$

where the matrices:

$$L^q(x)=\begin{bmatrix}\bar{l}^{q-1}(x)\\ \tilde{l}^{q}(x)\end{bmatrix},\qquad L^{q,e}(x)=\begin{bmatrix}\bar{l}^{q-1}(x) & \bar{c}^{q-1}(x)\\ 0 & \hat{c}^{q-1}(x)\\ \tilde{l}^{q}(x) & \tilde{c}^{q}(x)\end{bmatrix}$$

have ranks p_q, m_q, respectively ($p_q \leq m_q < p$). Then, there exists a nonsingular analytic $p\times p$ matrix $E^{q,1}(x)$ of the form:

$$E^{q,1}(x)=\begin{bmatrix}I_{m_{q-1}} & 0\\ R^q(x) & S^q(x)\end{bmatrix}$$

such that

$$E^{q,1}(x)L^{q,e}(x)=\begin{bmatrix}\bar{l}^{q-1}(x) & \bar{c}^{q-1}(x)\\ 0 & \hat{c}^{q-1}(x)\\ \hdashline l^q(x) & c^{q,1}(x)\\ 0 & c^{q,2}(x)\\ 0 & 0\end{bmatrix},$$

where $l^q(x)$, $c^{q,1}(x)$, $c^{q,2}(x)$ are matrices of dimensions $(p_q-p_{q-1})\times p$, $(p_q-p_{q-1})\times m$, $(m_q-m_{q-1}-p_q+p_{q-1})\times m$, respectively. The last $p-m_q$ rows of the preceding matrix are identically zero. Furthermore, there exists a $p\times p$ permutation matrix $E^{q,2}$ that rearranges the rows of $E^{q,1}(x)L^{q,e}(x)$ to obtain

$$E^{q,2}E^{q,1}(x)L^{q,e}(x)=\begin{bmatrix}\bar{l}^{q}(x) & \bar{c}^{q}(x)\\ 0 & \hat{c}^{q}(x)\\ 0 & 0\end{bmatrix},$$

where

$$\bar{c}^q(x)=\begin{bmatrix}\bar{c}^{q-1}(x)\\ c^{q,1}(x)\end{bmatrix},\qquad \hat{c}^q(x)=\begin{bmatrix}\hat{c}^{q-1}(x)\\ c^{q,2}(x)\end{bmatrix}$$

and the $p_q\times p$, $m_q\times(p+m)$ matrices:

$$\bar{l}^q(x)=\begin{bmatrix}\bar{l}^{q-1}(x)\\ l^q(x)\end{bmatrix},\qquad \begin{bmatrix}\bar{l}^{q}(x) & \bar{c}^{q}(x)\\ 0 & \hat{c}^{q}(x)\end{bmatrix}$$

have full row rank. Define $E^q(x)=E^{q,2}E^{q,1}(x)$.

Step 1. Premultiply the algebraic equations (Eq. 6) with the matrix $E^q(x)$ to obtain

$$0=\begin{bmatrix}\bar{k}^{q}(x)\\ \hat{k}^{q}(x)\\ \boldsymbol{k}^{q}(x)\end{bmatrix}+\begin{bmatrix}\bar{l}^{q}(x)\\ 0\\ 0\end{bmatrix}z+\begin{bmatrix}\bar{c}^{q}(x)\\ \hat{c}^{q}(x)\\ 0\end{bmatrix}u, \tag{7}$$

where $\bar{k}^q(x)$, $\hat{k}^q(x)$, $\boldsymbol{k}^q(x)$ are analytic vector fields of dimensions p_q, (m_q-p_q), $(p-m_q)$, respectively, and the last $p-p_q$ equations denote underlying algebraic constraints in x.

Step 2. Differentiate the constraints in x that are independent of u, that is, the last $p-m_q$ equations in Eq. 7, to obtain the following new set of algebraic equations:

$$0=\begin{bmatrix}\bar{k}^{q}(x)\\ \hat{k}^{q}(x)\\ \tilde{k}^{q+1}(x)\end{bmatrix}+\begin{bmatrix}\bar{l}^{q}(x)\\ 0\\ \tilde{l}^{q+1}(x)\end{bmatrix}z+\begin{bmatrix}\bar{c}^{q}(x)\\ \hat{c}^{q}(x)\\ \tilde{c}^{q+1}(x)\end{bmatrix}u, \tag{8}$$

where $\tilde{k}^{q+1}(x)$ is a vector field of dimension $p-m_q$ and $\tilde{l}^{q+1}(x)$, $\tilde{c}^{q+1}(x)$ are matrices of dimensions $(p-m_q)\times p$, $(p-m_q)\times m$, respectively, defined in a fashion analogous to that in Iteration 1.

Step 3. Evaluate the rank of the following matrices:

$$\operatorname{rank}\begin{bmatrix}\bar{l}^{q}(x)\\ \tilde{l}^{q+1}(x)\end{bmatrix}=p_{q+1},$$

$$\operatorname{rank}\begin{bmatrix}\bar{l}^{q}(x) & \bar{c}^{q}(x)\\ 0 & \hat{c}^{q}(x)\\ \tilde{l}^{q+1}(x) & \tilde{c}^{q+1}(x)\end{bmatrix}=m_{q+1},\qquad m_{q+1}\geq p_{q+1}.$$

If $m_{q+1}=p$, then stop, else repeat the preceding steps for the next iteration, starting with the algebraic equations in Eq. 8.

For a solvable DAE system of Eq. 1 with a finite index ν_d, the algorithmic procedure will converge in a finite number of iterations s with $p_1\leq p_2\leq\cdots\leq p_{s+1}\leq p$ and $m_2\leq m_3\leq\cdots\leq m_{s+1}=p$, $(m_i\geq p_i,\ \forall i>1)$. The algorithm identifies the following algebraic constraints among the differential variables x:

$$\boldsymbol{k}(x)=\begin{bmatrix}\boldsymbol{k}^1(x)\\ \vdots\\ \boldsymbol{k}^s(x)\end{bmatrix}=0, \tag{9}$$

which do not involve the inputs u. It can be shown, following an approach similar to that in Kumar and Daoutidis (1995a), that these constraints are linearly independent, that is, the gradient vector fields

$$\frac{d\boldsymbol{k}_i(x)}{dx}=\left[\frac{\partial \boldsymbol{k}_i}{\partial x_1}(x)\cdots\frac{\partial \boldsymbol{k}_i}{\partial x_n}(x)\right]$$

are linearly independent, where $k_i(x)$ is the ith component of the vector field $k(x)$. The iterative procedure also yields the following set of algebraic equations:

$$0=\begin{bmatrix}\bar{k}^s(x)\\ \hat{k}^s(x)\\ \tilde{k}^{s+1}(x)\end{bmatrix}+\begin{bmatrix}\bar{l}^s(x)\\ 0\\ \tilde{l}^{s+1}(x)\end{bmatrix}z+\begin{bmatrix}\bar{c}^s(x)\\ \hat{c}^s(x)\\ \tilde{c}^{s+1}(x)\end{bmatrix}u, \tag{10}$$

where the matrices:

$$L^{s+1}(x)=\begin{bmatrix}\bar{l}^s(x)\\ \tilde{l}^{s+1}(x)\end{bmatrix},\qquad L^{s+1,e}(x)=\begin{bmatrix}\bar{l}^s(x) & \bar{c}^s(x)\\ 0 & \hat{c}^s(x)\\ \tilde{l}^{s+1}(x) & \tilde{c}^{s+1}(x)\end{bmatrix}$$

have ranks p_{s+1} and $m_{s+1}=p$, respectively. Thus, there exists a nonsingular analytic $p\times p$ matrix $E^{s+1}(x)$ such that

$$E^{s+1}(x)L^{s+1,e}(x)=\begin{bmatrix}\bar{l}(x) & \bar{c}(x)\\ 0 & \hat{c}(x)\end{bmatrix},$$

where $\bar{c}(x)$, $\hat{c}(x)$ are matrices of dimensions $p_{s+1}\times m$, $(p-p_{s+1})\times m$, respectively, and the $p_{s+1}\times p$, $p\times(p+m)$ matrices:

$$\bar{l}(x)\begin{bmatrix}\bar{l}(x) & \bar{c}(x)\\ 0 & \hat{c}(x)\end{bmatrix}$$

have full row rank. Premultiplying the algebraic equations in Eq. 10 with the matrix $E^{s+1}(x)$, the following final set of algebraic equations is obtained:

$$0=\begin{bmatrix}\bar{k}(x)\\ \hat{k}(x)\end{bmatrix}+\begin{bmatrix}\bar{l}(x)\\ 0\end{bmatrix}z+\begin{bmatrix}\bar{c}(x)\\ \hat{c}(x)\end{bmatrix}u, \tag{11}$$

where $\bar{k}(x)$, $\hat{k}(x)$ are analytic vector fields of dimensions p_{s+1} and $p-p_{s+1}$, respectively. Thus, the algorithmic procedure yields the following new DAE system:

$$\begin{aligned}\dot{x}&=f(x)+b(x)z+g(x)u\\ 0&=\begin{bmatrix}\bar{k}(x)\\ \hat{k}(x)\end{bmatrix}+\begin{bmatrix}\bar{l}(x)\\ 0\end{bmatrix}z+\begin{bmatrix}\bar{c}(x)\\ \hat{c}(x)\end{bmatrix}u\\ y_i&=h_i(x),\qquad i=1,\ldots,m,\end{aligned} \tag{12}$$

where $x\in\mathfrak{X}:k(x)=0$. This DAE system (Eq. 12) is equivalent to the original DAE system in Eq. 1, that is, for consistent initial conditions ($x(0)$, $z(0)$) and smooth inputs $u(t)$, both systems have the same solution $x(t)$, $z(t)$. Moreover, note that the algebraic equations in Eq. 12 explicitly include the underlying constraints in x, $0=\hat{k}(x)+\hat{c}(x)u$ that involve the inputs u in an irreducible fashion, that is, $\hat{c}(x)$ has a full row rank $p-p_{s+1}$.

Remark 1. The choice of the matrices $E^q(x)$ in various iterations of the algorithm is not unique. However, following an approach similar to Hirschorn (1979), it can be shown that the ranks p_q and m_q in successive iterations of the algorithm are invariant under different choices of $E^q(x)$. Thus, the absence ($m_i=p_i$, $\forall i\geq 1$) or the presence ($m_i>p_i$, $\forall i\geq k$ for some $k\in[1, s+1]$) of the manipulated inputs u in the underlying constraints in x is an inherent characteristic of the DAE system of Eq. 1, and the system is correspondingly regular or nonregular.

Remark 2. For a regular DAE system of the form in Eq. 1, the proposed algorithmic procedure reduces to that in Kumar and Daoutidis (1995a), and it converges after exactly $s=\nu_d-1$ iterations. On the other hand, for nonregular DAE systems, the proposed algorithmic procedure will converge in s iterations, where in general s will be significantly less than ν_d-1.

The DAE system of Eq. 1, or equivalently of Eq. 12, is clearly *nonregular*, owing to the constraints $0=\hat{k}(x)+\hat{c}(x)u$ that involve the manipulated inputs u. In the next section, the DAE system of Eq. 12 will be *regularized* through feedback, to facilitate the derivation of state-space realizations that can be used as the basis for a feedback controller synthesis.

Dynamic Feedback Regularization

Consider the nonregular DAE system in Eq. 12 obtained through the proposed algorithmic procedure. It is desired to derive a state-feedback compensator such that the resulting feedback-modified DAE system with a new vector of inputs is regular. Note, however, that the DAE system in Eq. 12 still has a high index and the constraints $0=\hat{k}(x)+\hat{c}(x)u$ have to be differentiated at least once, to obtain a set of algebraic equations solvable in z. Thus, the solution for the algebraic variables z is a function of the differential variables x, the manipulated inputs u and *at least* one of their derivatives, that is, it has the form

$$z(t)=\varphi(x(t),u(t),u^{(1)}(t),\ldots), \tag{13}$$

where $u^{(i)}$ denotes the ith derivative of the manipulated input vector u. Thus, any causal feedback law for u must be independent of z. Furthermore, any static feedback of the form

$$u=\mathfrak{F}(x,v)$$

where $v\in\mathbb{R}^m$ is the new input vector ($(\partial\mathfrak{F}/\partial v)$ is nonsingular), will not regularize the DAE system of Eq. 12, since the constraints $0=\hat{k}(x)+\hat{c}(x)\mathfrak{F}(x,v)$ in the resulting modified system would still involve the inputs v.

These observations indicate the need for a *dynamic* feedback compensator to modify the DAE system in Eq. 12, in particular the constraints:

$$0=\hat{k}(x)=\hat{c}(x)u \tag{14}$$

that cause the nonregularity, such that the resulting system with new inputs is regular. A systematic derivation of the compensator involves the following two steps.

Step 1. Employ an input transformation:

$$u=M(x)\begin{bmatrix}\bar{u}_1\\ \bar{u}_2\end{bmatrix},$$

where $\bar{u}_1 \in \mathbb{R}^{(p-p_{s+1})}$, $\bar{u}_2 \in \mathbb{R}^{m-(p-p_{s+1})}$, to isolate the $p-p_{s+1}$ inputs $\bar{u}_1$ that appear in the constraints of Eq. 14 in a nonsingular fashion. This can be accomplished in a straightforward fashion, by choosing $M(x)$ as a nonsingular analytic $m \times m$ matrix such that

$$\begin{bmatrix} \bar{c}(x) \\ \hat{c}(x) \end{bmatrix} M(x) = \begin{bmatrix} \bar{c}_1(x) & \bar{c}_2(x) \\ \hat{c}_1(x) & 0 \end{bmatrix}, \tag{15}$$

where the $(p-p_{s+1})\times(p-p_{s+1})$ matrix $\hat{c}_1(x)$ is nonsingular. Under this input transformation, the constraints in Eq. 14 take the following form:

$$0 = \hat{k}(x) + \hat{c}_1(x)\bar{u}_1. \tag{16}$$

Step 2. Design a dynamic feedback compensator for the inputs $\bar{u}_1$ with the following general form:

$$\dot{w} = v_1$$
$$\bar{u}_1 = (\hat{c}_1(x))^{-1}[-\hat{k}(x) + \mathcal{C}(x,w)],$$

where $w \in \mathbb{R}^{p-p_{s+1}}$ is the vector of compensator states and $\mathcal{C}(x,w)$ is an analytic vector field of dimension $p-p_{s+1}$, to modify the constraints in Eq. 16 so that

(a) the resulting constraints:

$$0 = \mathcal{C}(x,w) \tag{17}$$

involve only the differential variables x and the compensator states w, and

(b) the new set of algebraic equations obtained after one differentiation of these constraints (Eq. 17) can be solved for the algebraic variables z.

From the fact that the feedback compensator must be independent of z, it is evident that upon differentiation of the modified constraints (Eq. 17) in Step 2b, the algebraic variables z can appear only through the differential variables x. Thus, the dynamic feedback compensator of Step 2, in particular $\mathcal{C}(x,w)$, must be designed to alter the dependence of the constraints in Eq. 17 on the variables x in a way such that the requirement in Step 2b is met. The existence of such a compensator is ensured by an important characteristic of solvable semi-explicit DAE systems of the form in Eq. 1 with a finite index ν_d, stated in the following Lemma (for a proof, see the Appendix).

Lemma 1. Consider a solvable semiexplicit DAE system described by:

$$\begin{aligned} \dot{x} &= f(x) + b(x)z + g(x)u(t) \\ 0 &= k(x) + l(x)z + c(x)u(t) \end{aligned} \tag{18}$$

with a finite index ν_d, where $x \in \mathcal{X} \subset \mathbb{R}^n$, $z \in \mathcal{Z} \subset \mathbb{R}^p$, $u(t) = [u_1(t) \cdots u_m(t)]^T$ is a vector of smooth inputs, $f(x), k(x)$ are analytic vector fields of dimensions n, p, respectively, and $b(x)$, $g(x)$, $l(x)$, $c(x)$ are analytic matrices of appropriate dimensions. Then, the $(n+p)\times p$ matrix

$$\begin{bmatrix} b(x) \\ l(x) \end{bmatrix} \tag{19}$$

has full column rank on $\mathcal{X}$.

Remark 3. For a linear analog of the DAE system in analog Eq. 18, described by

$$\begin{aligned} \dot{x} &= Ax + Bz + Gu \\ 0 &= Kx + Lz + Cu, \end{aligned} \tag{20}$$

where A, B, G, K, L, C are constant matrices of appropriate dimensions, the result of Lemma 1 follows directly (Kumar and Daoutidis, 1996) from the fact that for solvable DAE systems of the form in Eq. 20, the matrix pencil:

$$\mathcal{P} = \begin{bmatrix} \lambda I_n - A & -B \\ -K & -L \end{bmatrix}$$

is regular, that is, det $\mathcal{P} \not\equiv 0$.

The result of Lemma 1 implies that for the DAE system in Eq. 12, the matrix

$$\begin{bmatrix} b(x) \\ \bar{l}(x) \end{bmatrix}$$

has full rank p, and thus, there exists a matrix $S \in \mathbb{R}^{(p-p_{s+1})\times n}$ such that the following $p \times p$ matrix

$$\begin{bmatrix} Sb(x) \\ \bar{l}(x) \end{bmatrix} \tag{21}$$

is nonsingular. This fact implies that the desired goal of feedback regularization will be achieved if the constraint in Eq. 17, obtained through feedback modification, has the form

$$0 = \mathcal{C}(x,w) = Sx + w. \tag{22}$$

Theorem 1, which follows, provides an explicit representation of the requisite dynamic feedback regularizing compensator, derived along these lines.

Theorem 1. Consider a nonlinear semiexplicit DAE system of the form of Eq. 1, for which the proposed algorithmic procedure yields the equivalent DAE system of Eq. 12. Then the dynamic feedback compensator:

$$\dot{w} = v_1$$
$$u = M(x)\begin{bmatrix} (\hat{c}_1(x))^{-1}(-\hat{k}(x) + Sx + w) \\ 0 \end{bmatrix} + M(x)\begin{bmatrix} 0 \\ v_2 \end{bmatrix}, \tag{23}$$

where w, $v_1 \in \mathbb{R}^{(p-p_{s+1})}$, $v_2 \in \mathbb{R}^{m-(p-p_{s+1})}$ and the matrices S, $M(x)$ are chosen as in Eqs. 21 and 15, respectively, yields the modified DAE system:

$$
\begin{bmatrix}\dot{x}\\ \dot{w}\end{bmatrix}=\begin{bmatrix}\tilde{f}(x,w)\\ 0\end{bmatrix}+\begin{bmatrix}b(x)\\ 0\end{bmatrix}z+\begin{bmatrix}0 & \bar{g}_2(x)\\ I_{p-p_{s+1}} & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}
$$

$$
0=\begin{bmatrix}\tilde{k}(x,w)\\ Sx+w\end{bmatrix}+\begin{bmatrix}l(x)\\ 0\end{bmatrix}z+\begin{bmatrix}0 & \bar{c}_2(x)\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}
$$

$$
y_i=h_i(x),\qquad i=1,\ldots,m \tag{24}
$$

where

$$
\tilde{f}(x,w)=f(x)+\bar{g}_1(x)\gamma(x)+\bar{g}_1(x)(\hat{c}_1(x))^{-1}w
$$

$$
\tilde{k}(x,w)=\bar{k}(x)+\bar{c}_1(x)\gamma(x)+\bar{c}_1(x)(\hat{c}_1(x))^{-1}w
$$

$$
\gamma(x)=(\hat{c}_1(x))^{-1}\{-\hat{k}(x)+Sx\}
$$

$$
[\bar{g}_1(x)\ \bar{g}_2(x)]=g(x)M(x)
$$

and $x\in\mathfrak{X}$: $k(x)=0$. The DAE system of Eq. 24 with an extended vector of differential variables $\bar{x}=[x^T\quad w^T]^T$ and a new vector of inputs $v=[v_1^T\quad v_2^T]^T\in\mathbb{R}^m$ is regular.

Proof. For the DAE system in Eq. 12, the dynamic feedback compensator of Eq. 23 directly yields the extended DAE system in Eq. 24, where the last $p-p_{s+1}$ algebraic equations, $Sx+w=0$, denote constraints among the differential variables $\bar{x}$ that are independent of the inputs v. Differentiating these constraints once, the following set of algebraic equations is obtained:

$$
0=\begin{bmatrix}\tilde{k}(x,w)\\ S\tilde{f}(x,w)\end{bmatrix}+\begin{bmatrix}\bar{l}(x)\\ Sb(x)\end{bmatrix}z+\begin{bmatrix}0 & \bar{c}_2(x)\\ I_{p-p_{s+1}} & S\bar{g}_2(x)\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}. \tag{25}
$$

Clearly, by the choice of the matrix S as in Eq. 21, the coefficient matrix for z in the previous algebraic equations, that is,

$$
\begin{bmatrix}\bar{l}(x)\\ Sb(x)\end{bmatrix}
$$

is nonsingular. Thus, the DAE system in Eq. 24 is solvable, with index $\nu_d=2$ and the unique smooth solution for z:

$$
z=-\begin{bmatrix}\bar{l}(x)\\ Sb(x)\end{bmatrix}^{-1}\left\{\begin{bmatrix}\tilde{k}(x,w)\\ S\tilde{f}(x,w)\end{bmatrix}+\begin{bmatrix}0 & \bar{c}_2(x)\\ I_{p-p_{s+1}} & S\bar{g}_2(x)\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}\right\}
$$

$$
=R(x,w)+S_1(x)v_1+S_2(x)v_2. \tag{26}
$$

Moreover, the fact that Eq. 25 is solvable in z implies that there are no additional constraints in the differential variables $\bar{x}$, and the constraints $k(x)=0$, $Sx+w=0$, which are independent of the inputs v, specify an $n-\sum_{i=1}^{s}(p-m_i)$-dimensional subspace:

$$
\mathfrak{M}=\left\{(x,w)\in\mathfrak{X}\times\mathbb{R}^{(p-p_{s+1})}\ :\ \begin{matrix}k(x)=0\\ Sx+w=0\end{matrix}\right\}\subset\mathbb{R}^{n+(p-p_{s+1})}. \tag{27}
$$

It can be verified that for any initial condition $\bar{x}(0)\in\mathfrak{M}$ and under the unique solution of z in Eq. 26, the differential variables $\bar{x}(t)$ are constrained to evolve in the subspace $\mathfrak{M}$ for any smooth input $v(t)$. Thus, $\mathfrak{M}$ is the constrained state space of the DAE system in Eq. 24, which is invariant under any control law for the inputs v. This establishes that the feedback modified DAE system of Eq. 24 is regular.

Remark 4. Clearly, the choice of the constant matrix S in Eq. 21 is not unique. For the purpose of regularization, the matrix S can always be chosen to be a simple permutation matrix that selects $p-p_{s+1}$ rows of the matrix $b(x)$, which, together with the matrix $\bar{l}(x)$, comprise a nonsingular $p\times p$ matrix. However, one has the flexibility to choose a more general matrix S, which directly affects the constrained state space $\mathfrak{M}$, and hence the state-space realization of the feedback regularized DAE system (Eq. 24), to attain additional objectives besides regularization.

The overall feedback regularization procedure for a nonregular DAE system of Eq. 1 is shown in Figure 1.

State-Space Realizations and Controller Synthesis

For the feedback regularized DAE system in Eq. 24, the solution for z (Eq. 26) and the specification of the constrained state space (Eq. 27) allow the derivation of a state-space realization, described by a set of explicit differential equations for $\bar{x}$ on the state space $\mathfrak{M}$. The state-space realization is given in Proposition 1, which follows. For a proof, see the Appendix.

Proposition 1. Consider a nonlinear semiexplicit DAE system of the form in Eq. 1, for which the proposed algorithmic procedure converges after s iterations and the dynamic feedback compensator of Theorem 1 yields the regularized extended DAE system in Eq. 24. Then, the dynamic system:

$$
\dot{\bar{x}}=\bar{f}(\bar{x})+\bar{g}(\bar{x})v
$$

$$
y_i=\bar{h}_i(\bar{x}),\qquad i=1,\ \ldots,\ m \tag{28}
$$

is a state-space realization of the feedback regularized DAE system, where $\bar{x}=[x^T\ w^T]^T\in\mathfrak{M}$ is the extended state vector, $v\in\mathbb{R}^m$ is the new input vector defined in Theorem 1, and

$$
\bar{f}(\bar{x})=\begin{bmatrix}\tilde{f}(x,w)+b(x)R(x,w)\\ 0\end{bmatrix},
$$

$$
\bar{g}(\bar{x})=\begin{bmatrix}b(x)S_1(x) & \bar{g}_2(x)+b(x)S_2(x)\\ I_{p-p_{s+1}} & 0\end{bmatrix},
$$

$$
\bar{h}_i(\bar{x})=h_i(x).
$$

In view of the dimension $\kappa=n-\sum_{i=1}^{s}(p-m_i)$ of the constrained state space $\mathfrak{M}$ (Eq. 27) for the state variables $\bar{x}$, the state-space realization of Eq. 28 is clearly not of minimal order. A minimal-order realization can be obtained in a suitably transformed set of coordinates. More specifically, using a nonlinear coordinate transformation of the form:

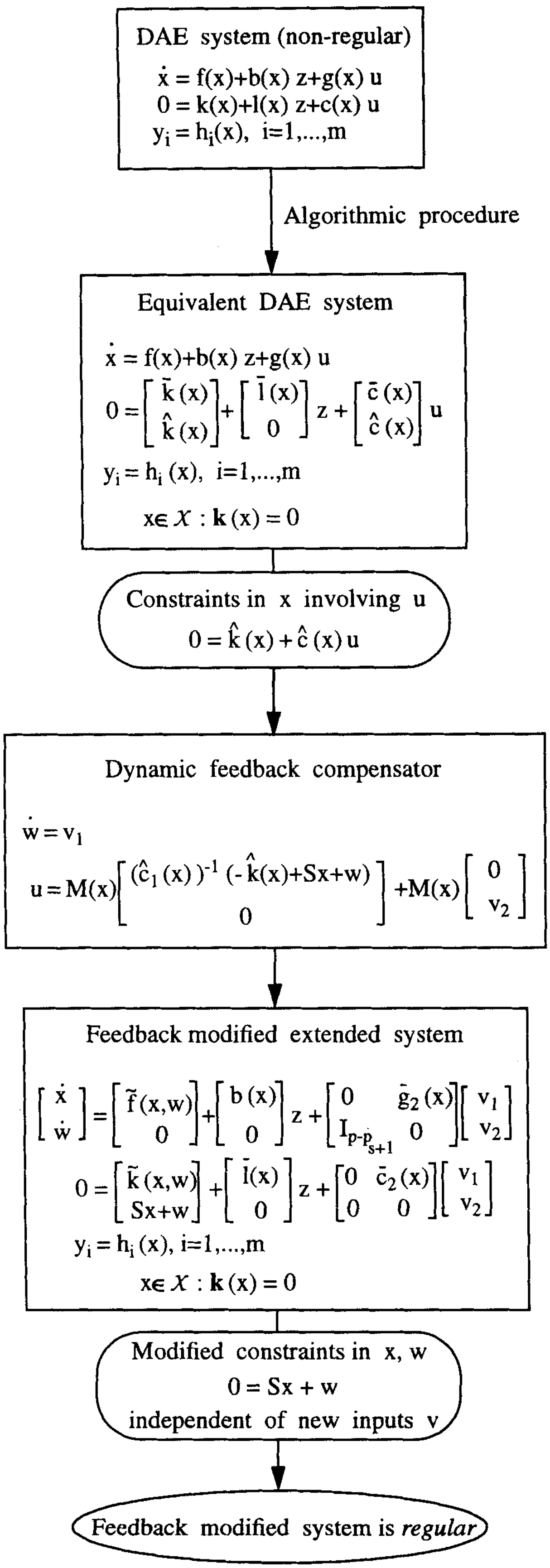

Figure 1. Steps in feedback regularization of a nonregular DAE system.

$$\zeta=\begin{bmatrix}\zeta_1^{(0)}\\ \vdots\\ \zeta_\kappa^{(0)}\\ \zeta^{(1)}\\ \zeta^{(2)}\end{bmatrix}=T(\bar{x})=\begin{bmatrix}\psi_1(x)\\ \vdots\\ \psi_\kappa(x)\\ \boldsymbol{k}(x)\\ Sx+w\end{bmatrix}, \tag{29}$$

where $\zeta^{(0)}\in\mathbb{R}^{\kappa}$, $\zeta^{(1)}\in\mathbb{R}^{n-\kappa}$, $\zeta^{(2)}\in\mathbb{R}^{p-p_{s+1}}$, and $\psi_1(x),\dots,\psi_\kappa(x)$ are scalar fields chosen such that $\zeta=T(\bar{x})$ in Eq. 29 is a local diffeomorphism, and eliminating the states $\zeta^{(1)}$, $\zeta^{(2)}$ that are identically zero on the state space $\mathfrak{M}$, a state-space realization of dimension κ can be obtained.

Remark 5. For a class of DAE systems where the algebraic equations and their derivatives directly yield the constraints in the differential variables, an approach based on dynamic extension has been proposed by Yim and Singh (1993). In this approach, the manipulated inputs and their derivatives that appear in these constraints are included in an extended state vector to obtain a state-space realization and design feedback controllers. However, this controller design method leads to a state-space realization and a dynamic controller of inordinately high order, as opposed to the feedback regularization method proposed in this work.

A state-feedback controller synthesis problem for the DAE system of Eq. 1 can now be formulated and solved on the basis of the state-space realization (Eq. 28) of the feedback-regularized system (Eq. 24). To this end, the notions of equilibrium points, stability of solutions, zero dynamics and characterization of minimum-phase behavior, and relative orders between the outputs and inputs can be introduced for the regularized DAE system in Eq. 24, on the basis of the state-space realization of Eq. 28. Specifically, for the feedback regularized DAE system of Eq. 24, we define the relative order r_i of the controlled output y_i with respect to the manipulated input vector v, as the minimum integer such that:

$$\left[L_{\bar{g}_1}L_{\bar{f}}^{r_i-1}\bar{h}_i(\bar{x})\quad L_{\bar{g}_2}L_{\bar{f}}^{r_i-1}\bar{h}_i(\bar{x})\cdots L_{\bar{g}_m}L_{\bar{f}}^{r_i-1}\bar{h}_i(\bar{x})\right]\not\equiv[0\,0\,\cdots\,0]$$

for $\bar{x}\in X\subset\mathfrak{M}$, where X is an open connected set containing the equilibrium point of interest. It is assumed that a finite relative order r_i exists for every controlled output y_i, since it is necessary for output controllability. The matrix:

$$C(\bar{x})=\begin{bmatrix}L_{\bar{g}_1}L_{\bar{f}}^{r_1-1}\bar{h}_1(\bar{x}) & \cdots & L_{\bar{g}_m}L_{\bar{f}}^{r_1-1}\bar{h}_1(\bar{x})\\ \vdots & & \vdots\\ L_{\bar{g}_1}L_{\bar{f}}^{r_m-1}\bar{h}_m(\bar{x}) & \cdots & L_{\bar{g}_m}L_{\bar{f}}^{r_m-1}\bar{h}_m(\bar{x})\end{bmatrix} \tag{30}$$

known as the characteristic matrix also will be assumed to be nonsingular on X, for simplicity. Finally, it is assumed that the zero-output-constrained dynamics (or simply zero dynamics) for the system of Eq. 28 is locally asymptotically stable, that is, the system is minimum phase.

Consider the DAE system of Eq. 1, for which the proposed regularization scheme yields the regular extended DAE system of Eq. 24 with the state-space realization of Eq. 28, a nonsingular characteristic matrix $C(\bar{x})$, and a locally asymptotically stable zero dynamics. For such a system, it is desired to derive a state-feedback controller that enforces the following closed-loop characteristics:

1. induce an input/output behavior described by

$$\sum_{i=1}^{m}\sum_{j=0}^{r_i}\beta_{ij}\frac{d^j y_i}{dt^j}=\tilde{v}, \tag{31}$$

where $\tilde{v}=[\tilde{v}_1 \cdots \tilde{v}_m]^T$ is the vector of external reference inputs and $\beta_{ij}=[\beta_{ij}^1 \cdots \beta_{ij}^m]^T$ are vectors of adjustable parameters, and

2. ensure input/output and internal stability,

subject to the underlying constraints imposed by the algebraic equations.

A solution to this synthesis problem will be derived on the basis of the state-space realization (Eq. 28) of the feedback-regularized DAE system (Eq. 24). More specifically, a static state-feedback controller will be derived for the system in Eq. 28 to induce the input/output behavior of Eq. 31 in the closed-loop system. The resulting controller, together with the dynamic feedback regularizing compensator of Theorem 1, will comprise the overall *dynamic* state-feedback controller for the DAE system of Eq. 1, as shown in Figure 2. The main result of this section is stated in Theorem 2, which follows (for a proof, see the Appendix).

Theorem 2. Consider a DAE system of the form of Eq. 1, for which the proposed algorithmic procedure and dynamic feedback regularizing compensator of Theorem 1 yield the regular extended DAE system of Eq. 24, with the state-space realization of Eq. 28 and a characteristic matrix $C(\bar{x})$, where det $C(\bar{x})\neq 0$, $\forall \bar{x}\in X$. Then, for the DAE system in Eq. 1, the dynamic state feedback law:

$$\dot{w}=v_1$$

$$\begin{bmatrix} v_1\\ v_2\end{bmatrix}=\{[\beta_{1r_1}\cdots\beta_{mr_m}]C(x,w)\}^{-1}\times\left[\tilde{v}-\sum_{i=1}^{m}\sum_{j=0}^{r_i}\beta_{ij}L_f^j\bar{h}_i(x,w)\right]$$

$$u=M(x)\begin{bmatrix}(\hat{c}_1(x))^{-1}(-\hat{k}(x)+Sx+w)\\ 0\end{bmatrix}+M(x)\begin{bmatrix}0\\ v_2\end{bmatrix} \tag{32}$$

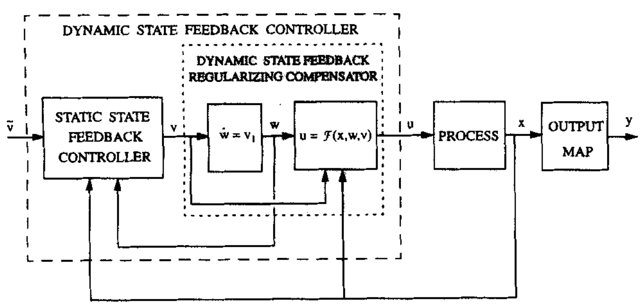

Figure 2. Overall structure of dynamic state feedback controller.

induces the input/output behavior:

$$\sum_{i=1}^{m}\sum_{j=0}^{r_i}\beta_{ij}\frac{d^j y_i}{dt^j}=\tilde{v}$$

subject to the algebraic constraints that describe the constrained subspace in Eq. 27.

Under a proper choice of the adjustable parameters β_{ij}^k that ensure the bounded-input bounded-output (BIBO) stability of the input/output behavior described in Eq. 31, and the assumption of local asymptotic stability of the zero dynamics of the system in Eq. 28, it can be verified that the closed-loop system is stable. Furthermore, a suitable linear error feedback controller with integral action can be incorporated around the linearized $\tilde{v}-y$ system, to induce the following closed-loop input/output behavior in the nominal system

$$y+\sum_{i=1}^{m}\sum_{j=1}^{r_i}\gamma_{ij}\frac{d^j y_i}{dt^j}=y_{sp} \tag{33}$$

and asymptotically reject the effects of small unmeasured disturbances and modeling errors (Daoutidis and Kravaris, 1994). In the input/output response requested in Eq. 33, $y=[y_1\cdots y_m]^T$, $y_{sp}=[y_{sp1}\cdots y_{spm}]^T$ are the output and setpoint vectors, while $\gamma_{ij}=[\gamma_{ij}^1\cdots\gamma_{ij}^m]^T$ are vectors of adjustable parameters.

Application of Control Methodology

High-index DAE systems arise naturally in the dynamic modeling of a wide variety of chemical processes, such as multiphase processes with fast mass transfer and reactors with fast reversible reactions. In these systems, the algebraic equations that are singular in nature and thus lead to a high index, are obtained by ignoring certain fast and stable dynamics and employing the corresponding pseudo-steady-state approximations instead (phase equilibrium, reaction equilibrium, etc.). Detailed models of such two-time-scale processes, incorporating both the slow and the fast dynamics (e.g., including the explicit correlations for the fast mass transfer, and the kinetic rate expressions for the forward and backward reactions in the case of fast reversible reactions), are typically given by index-one DAE systems. While such models can be used for dynamic simulation using appropriate methods for stiff ODEs, they are not suitable for controller design purposes. More specifically, it is well-documented (see, e.g, Christofides and Daoutidis, 1996; Kokotovic et al., 1986) that standard inversion-based controllers designed directly on the basis of such models, without accounting for the time-scale multiplicity, are often ill-conditioned, and they lead to closed-loop instability for slightly nonminimum phase systems. Standard controller design approaches that overcome these problems involve (1) stabilizing the fast dynamics, if they are unstable, and (2) designing feedback controllers to achieve desired closed-loop objectives on the basis of a quasi-steady-state model for the slow dynamics of the process, obtained by ignoring the stable fast dynamics. Thus, for the purpose of designing well-conditioned controllers, it is imperative to use

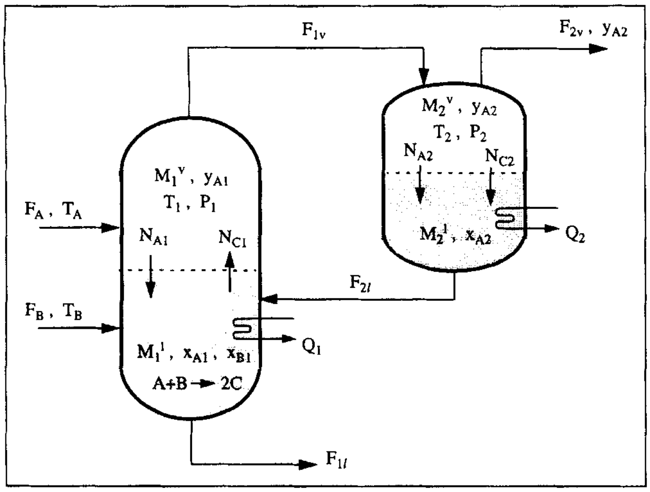

Figure 3. Two-phase reactor and a condenser with recycle.

the models for the slow dynamics of such processes, which are given by high-index DAE systems.

In the remaining part of this section, we apply the developed control methodology to an interconnection of a two-phase reactor and a condenser with liquid recycle, where the slow dynamics of the process are described by an index-three DAE system that is nonregular.

Process description and modeling

Consider the process in Figure 3, comprised of a two-phase (vapor/liquid) reactor and a condenser. Reactants A and B are fed to the reactor at molar flow rates F_A, F_B and temperatures T_A, T_B in the vapor and liquid phases, respectively. Reactant A diffuses into the liquid phase at a rate N_{A1}, where the exothermic reaction

$$A + B \rightarrow 2C$$

occurs. Product C diffuses into the vapor phase at a rate N_{C1}, while reactant B is assumed to be nonvolatile. It is assumed that the interphase mass-transfer resistance is negligible and the reaction rate in the bulk liquid phase is described by the following Arrhenius relation:

$$r_A = k_{10} \exp\left(\frac{-E_a}{RT_1}\right) M_1^l \rho x_{A1} x_{B1},$$

where r_A is the rate of consumption of reactant A (or equivalently B) at the temperature T_1; k_{10}, E_a are the preexponential factor and activation energy; M_1^l is the liquid molar holdup in the reactor; ρ is the molar density of the liquid; and x_{A1}, x_{B1} are the mole fractions of A and B in the liquid phase, respectively. For simplicity, it is assumed that the molar heat capacity c_p, density ρ, and latent heat of vaporization ΔH^v are constant and equal for all species, and the liquid and vapor phases are ideal mixtures. The liquid stream from the reactor is withdrawn at a constant flow rate F_{1l}, while the vapor stream at a flow rate F_{1v} enters the condenser. The vapor is cooled in the condenser to a temperature T_2, to enhance the purity of the product by removing the reactant A into the liquid phase (the relative volatility of C with respect to A increases with decreasing temperature). The liquid phase in the condenser, rich in reactant A, is recycled to the reactor at a flow rate F_{2l}, while the product vapor phase exits the condenser at a flow rate F_{2v} and a composition y_{A2}.

A detailed dynamic model of the process, which explicitly includes the modeling equations for the vapor holdup in the reactor and condenser, is given by a DAE system. The differential equations include the overall and component mole balances in the liquid and vapor phases and the overall enthalpy balances, in the reactor and the condenser. The algebraic equations, on the other hand, include the constitutive relations for the interphase mass-transfer rates, the ideal gas equations for the vapor holdups, and the pressure-drop correlation. The overall DAE model for the process is as follows:

$$\dot{M}_1^l = F_B - F_{1l} + F_{2l} + N_{A1} - N_{C1} \tag{34}$$

$$\dot{x}_{A1} = \left(\frac{1}{M_1^l}\right)[-F_B x_{A1} + F_{2l}(x_{A2} - x_{A1}) + N_{A1}(1 - x_{A1}) + N_{C1} x_{A1} - r_A] \tag{35}$$

$$\dot{x}_{B1} = \left(\frac{1}{M_l^l}\right)[F_B(1 - x_{B1}) - F_{2l} x_{B1} - N_{A1} x_{B1} + N_{C1} x_{B1} - r_A] \tag{36}$$

$$\dot{M}_1^v = F_A - F_{1v} - N_{A1} + N_{C1} \tag{37}$$

$$\dot{y}_{A1} = \left(\frac{1}{M_1^v}\right)[F_A(1 - y_{A1}) - N_{A1}(1 - y_{A1}) - N_{C1} y_{A1}] \tag{38}$$

$$\dot{T}_1 = \left(\frac{1}{M_1^l + M_1^v}\right)\Bigg[F_A(T_A - T_1) + F_B(T_B - T_1) + F_{2l}(T_2 - T_1) + (N_{A1} - N_{C1})\frac{\Delta H^v}{c_p} - \frac{Q_1}{c_p} + r_A\left(\frac{-\Delta H_r}{c_p}\right)\Bigg] \tag{39}$$

$$\dot{M}_2^l = N_{A2} + N_{C2} - F_{2l} \tag{40}$$

$$\dot{x}_{A2} = \left(\frac{1}{M_2^l}\right)[N_{A2}(1 - x_{A2}) - N_{C2} x_{A2}] \tag{41}$$

$$\dot{M}_2^v = F_{1v} - F_{2v} - N_{A2} - N_{C2} \tag{42}$$

$$\dot{y}_{A2} = \left(\frac{1}{M_2^v}\right)[F_{1v}(y_{A1} - y_{A2}) - N_{A2}(1 - y_{A2}) + N_{C2} y_{A2}] \tag{43}$$

$$\dot{T}_2 = \left(\frac{1}{M_2^l + M_2^v}\right)\Bigg[F_{1v}(T_1 - T_2) + (N_{A2} + N_{C2})\frac{\Delta H^v}{c_p} - \frac{Q_2}{c_p}\Bigg] \tag{44}$$

$$0 = N_{A1} - k_A a(y_{A1} - y_{A1}^*)\frac{M_1^l}{\rho} \tag{45}$$

$$0 = N_{C1} - k_C a(y^*_{C1} - (1 - y_{A1}))\frac{M^l_1}{\rho} \tag{46}$$

$$0 = N_{A2} - k_A a(y_{A2} - y^*_{A2})\frac{M^l_2}{\rho} \tag{47}$$

$$0 = N_{C2} - k_C a(1 - y_{A2} - y^*_{C2})\frac{M^l_2}{\rho} \tag{48}$$

$$0 = P_1\left(V_{1T} - \frac{M^l_1}{\rho}\right) - M^v_1 RT_1 \tag{49}$$

$$0 = P_2\left(V_{2T} - \frac{M^l_2}{\rho}\right) - M^v_2 RT_2 \tag{50}$$

$$0 = P_1 - P_2 - \frac{1}{0.09}(F_{1v})^{7/4}. \tag{51}$$

In these equations, k_A, k_C denote the overall mass-transfer coefficients for A and C, and a denotes the interfacial area per unit liquid holdup volume. Moreover, y^*_{A1}, y^*_{C1}, y^*_{A2}, y^*_{C2} denote compositions of the vapor phase at equilibrium with the liquid phase, and are given by the ideal phase equilibrium (Raoult's law) relations:

$$0 = P_1 y^*_{A1} - P^s_{A1} x_{A1} \tag{52}$$

$$0 = P_1 y^*_{C1} - P^s_{C1}(1 - x_{A1} - x_{B1}) \tag{53}$$

$$0 = P_2 y^*_{A2} - P^s_{A2} x_{A2} \tag{54}$$

$$0 = P_2 y^*_{C2} - P^s_{C2}(1 - x_{A2}), \tag{55}$$

where P^s_{Ai}, P^s_{Ci} are the saturation vapor pressures of A and C in the reactor ($i = 1$) and the condenser ($i = 2$), given by the Antoine relations:

$$P^s_{Ai} = \exp\left(25.1 - \frac{3{,}400}{T_i + 20}\right)$$

$$P^s_{Ci} = \exp\left(27.3 - \frac{4{,}100}{T_i + 70}\right).$$

For sufficiently large values of $k_A a$ and $k_C a$, that is, small mass-transfer resistances, the interphase mass transfer in the reactor and condenser are fast and stable, and the liquid and vapor phases are close to equilibrium. Furthermore, the dynamics of the pressure P_1 in the reactor is also fast, albeit unstable (see the simulations later in this section). The fast pressure dynamics can be easily stabilized at a desired level P^*, by a proportional feedback controller:

$$F_{2v} = F_{2v,\mathrm{nom}} - K(P^* - P_1) \tag{56}$$

using the product-vapor flow rate F_{2v} from the condenser as the manipulated input. In the preceding equation, $F_{2v,\mathrm{nom}}$ refers to the nominal steady-state value of F_{2v} and K is the controller gain. A description and nominal values of the process variables and parameters is included in Table 1.

The detailed DAE model (Eqs. 34–56) of the overall process with stable fast dynamics associated with the fast interphase mass transfer, and stabilized fast dynamics for the reactor pressure P_1, has an index-one and can be easily reduced to a standard ODE model. An input/output linearizing feedback controller can be derived on the basis of the ODE model, to control the outputs:

$$y_1 = T_1, \qquad y_2 = y_{A2}, \qquad y_3 = M^l_2$$

using the manipulated inputs:

$$u_1 = Q_1, \qquad u_2 = Q_2, \qquad u_3 = F_{2l}.$$

However, such a controller ignores the inherent time-scale multiplicity in the process and is ill-conditioned, that is, the control action is highly sensitive to small modeling/measurement errors since the effect of these errors is magnified through the large process parameters, for example, the large mass-transfer coefficients, which appear explicitly in the control law. In fact, in our simulations, the closed-loop system was found to be unstable at the nominal operating point. The corresponding profiles for the controlled outputs and manipulated inputs are shown in Figure 4. This closed-loop instability indicates a nonminimum phase behavior of the two-time-scale process. In particular, the product composition y_{A2} exhibits an inverse response for step changes in the condenser heat duty Q_2 and the recycle flow rate F_{2l}, as shown in Figure 5. This nonminimum phase behavior arises from the fast dynamics in the initial boundary layer, specifically the fast mass transfer, and such two-time-scale systems are said to be slightly nonminimum phase. The controller ill-conditioning and the closed-loop instability due to the slightly nonminimum phase behavior motivate the design of well-conditioned controllers on the basis of quasi-steady-state models that ignore the stable fast dynamics.

If the fast and stable dynamics associated with the mass transfer is ignored, that is, the explicit mass-transfer correlations (Eqs. 45–48) are replaced by the pseudo-steady-state conditions of phase equilibrium:

$$0 = P_1 y_{A1} - P^s_{A1} x_{A1} \tag{57}$$

$$0 = P_1(1 - y_{A1}) - P^s_{C1}(1 - x_{A1} - x_{B1}) \tag{58}$$

$$0 = P_2 y_{A2} - P^s_{A2} x_{A2} \tag{59}$$

$$0 = P_2(1 - y_{A2}) - P^s_{C2}(1 - x_{A2}), \tag{60}$$

while the reactor pressure dynamics governed by the control equation of Eq. 56 is retained, the resulting DAE system has an index-two and is regular. A feedback controller can be designed on the basis of this index-two model, following the approach of Kumar and Daoutidis (1995a). While the closed-loop system under this controller is stable, the controller is still ill-conditioned (see the subsection on controller synthesis and performance later in this section), implying the need to ignore the stabilized fast dynamics of the reactor pressure P_1, as well to obtain well-conditioned controllers.

A DAE model describing only the slow dynamics of the process is obtained by ignoring the fast and stable modes as-

Table 1. Description of Process Variables and Parameters and Their Nominal Values

Variable	Description	Nominal Value
a	Interfacial mass-transfer area/unit liquid holdup vol. (m^2/m^3)	1,000
c_p	Molar heat capacity (J/mol·K)	80.0
E_a	Activation energy in Arrhenius rate expression (kJ/mol)	110.0
F_A	Inlet molar flow rate of reactant A (mol/s)	99.84
F_B	Inlet molar flow rate of reactant B (mol/s)	52.0
F_{1l}	Molar flow rate of liquid stream from reactor (mol/s)	10.0
F_{2l}	Molar flow rate of liquid recycle from condenser to reactor (mol/s)	72.19
F_{1v}	Molar flow rate of vapor stream from reactor to condenser (mol/s)	214.03
F_{2v}	Molar flow rate of product vapor from condenser (mol/s)	141.84
K	Proportional gain of pressure controller (mol/s atm)	10.0
k_{10}	Preexponential factor (m^3/mol·s)	$2.88e+11$
k_A	Overall mass-transfer coeff. for A (mol/m^2·s)	20
k_C	Overall mass-transfer coeff. for C (mol/m^2·s)	30
M_1^l	Liquid molar holdup in reactor (kmol)	14.52
M_2^l	Liquid molar holdup in condenser (kmol)	15.0
M_1^v	Vapor molar holdup in reactor (kmol)	3.75
M_2^v	Vapor molar holdup in condenser (kmol)	3.90
P_1	Pressure in reactor (atm)	50.0
P_2	Pressure in condenser (atm)	48.69
P^*	Setpoint for reactor pressure (atm)	50.0
Q_1	Heat output from reactor (kW)	863.68
Q_2	Heat output from condenser (kW)	1,164.39
T_A	Temperature of feed A (K)	315.0
T_B	Temperature of feed B (K)	300.0
T_1	Temperature in reactor (K)	330.0
T_2	Temperature in condenser (K)	304.16
V_{1T}	Volume of reactor (m^3)	3.0
V_{2T}	Volume of condenser (m^3)	3.0
x_{A1}	Mole fraction of A in liquid phase in reactor	0.49
x_{B1}	Mole fraction of B in liquid phase in reactor	0.40
x_{A2}	Mole fraction of A in liquid phase in condenser	0.74
y_{A1}	Mole fraction of A in vapor phase in reactor	0.47
y_{A2}	Mole fraction of A in vapor phase in condenser	0.33
ΔH_r	Heat of reaction (kJ/mol)	−50.0
ΔH^v	Latent heta of vaporization (kJ/mol)	10.0
ρ	Liquid molar density (kmol/m^3)	15.0

sociated with the interphase mass transfer *and* the reactor pressure dynamics. More specifically, the explicit correlations for the mass-transfer rates in Eqs. 45–48 are replaced by the phase-equilibrium relations in Eqs. 57–60, and the control equation for the reactor pressure P_1 in Eq. 56 is replaced by the corresponding pseudo-steady-state condition:

$$0 = P_1 - P^*. \tag{61}$$

The differential variables in the DAE system include $M_1^l, x_{A1}, x_{B1}, M_1^v, y_{A1}, T_1, M_2^l, x_{A2}, M_2^v, y_{A2}, T_2$, while the algebraic variables include $P_1, N_{A1}, N_{C1}, F_{1v}, P_2, N_{A2}, N_{C2}, F_{2V}$. However, the algebraic variable F_{1v} appears in a nonlinear fashion in the pressure-drop correlation (Eq. 51). In view of this fact, the following dynamic extension:

$$\dot{F}_{1v} = \overline{F}_{1v} \tag{62}$$

is employed, where $\overline{F}_{1v}$ replaces F_{1v} as an algebraic variable and F_{1v} is included in an extended vector of differential variables. The resulting DAE system, with the differential equations in Eqs. 34–44 and 62, the algebraic equations in Eqs. 57–60, 49–51, and 61, the differential variables:

$$x_1 = M_1^l, \quad x_2 = x_{A1}, \quad x_3 = x_{B1}, \quad x_4 = M_1^v,$$
$$x_5 = y_{A1}, \quad x_6 = T_1 \quad x_7 = M_2^l, \quad x_8 = x_{A2},$$
$$x_9 = M_2^v, \quad x_{10} = y_{A2}, \quad x_{11} = T_2, \quad x_{12} = F_{1v}$$

and the algebraic variables:

$$z_1 = N_{A1}, \quad z_2 = N_{C1}, \quad z_3 = \overline{F}_{1v}, \quad z_4 = N_{A2},$$
$$z_5 = N_{C2}, \quad z_6 = F_{2v}, \quad z_7 = P_1, \quad z_8 = P_2$$

is in the form of Eq. 1, and has an index $\nu_d = 3$.

In what follows, a controller is derived on the basis of the index-three DAE model of the process. The algorithmic procedure converges after one iteration with $p_2 = 7$ ($< p$) and $m_2 = p$, implying the presence of a constraint in x that involves the manipulated inputs u, that is, the DAE system is

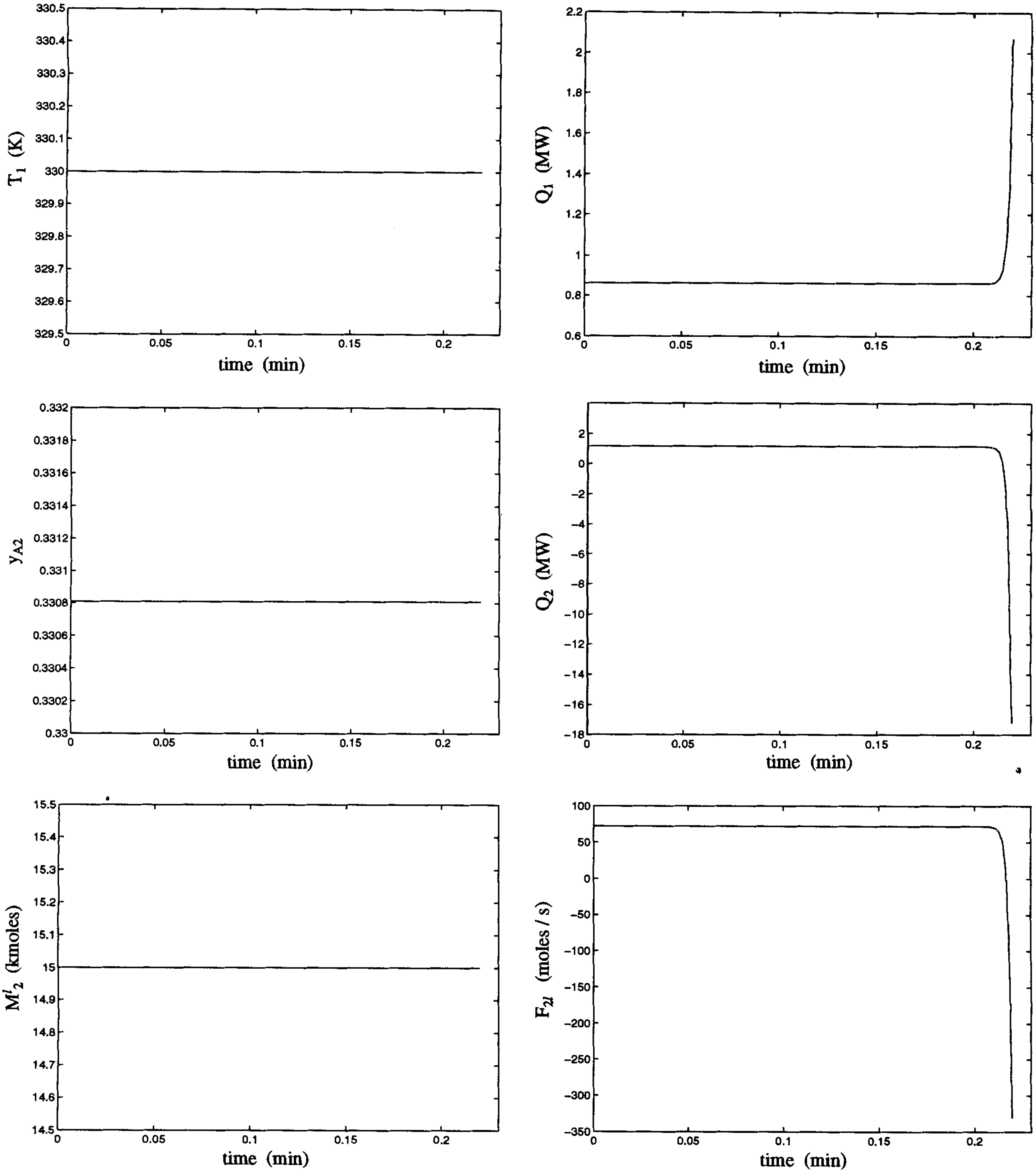

Figure 4. Closed-loop profiles of controlled outputs and manipulated inputs in nominal process, under controller based on the detailed index-one DAE model.

nonregular. The DAE system is regularized through the dynamic state-feedback compensator of Theorem 1, and an input/output linearizing controller of Theorem 2 is then derived on the basis of a state-space realization of the regularized system. It is shown through simulations that the controller performs satisfactorily and is well-conditioned.

Algorithmic procedure

Iteration 1

Step 1. Consider the algebraic equations for the DAE system (Eqs. 57–60, 49–51, and 61). They involve only two algebraic variables $z_7 = P_1$ and $z_8 = P_2$, that is, $p_1 = 2$. More-

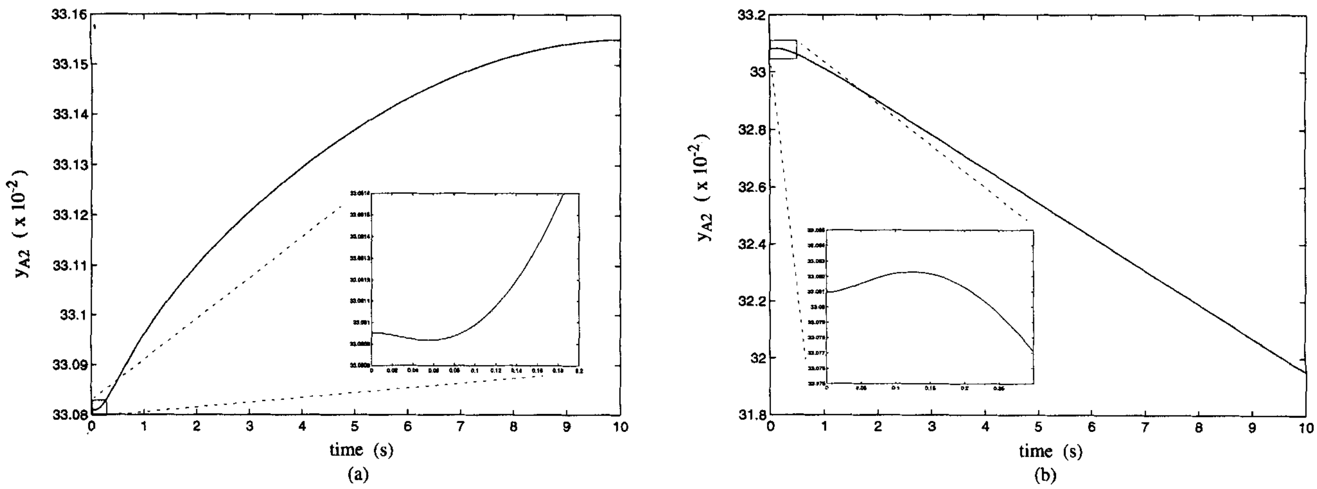

Figure 5. Open-loop profiles of y_{A2} showing inverse response for a step change in (a) F_{2l}, and (b) Q_2.

over, since the algebraic equations do not involve any of the manipulated inputs, the augmented matrix has a rank $m_1 = p_1$. Premultiply these equations by the nonsingular matrix:

$$E^1(x) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & e_{26} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -y_{A1} \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & e_{55} & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & e_{66} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & e_{76} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_{86} & 1 & -1 \end{bmatrix},$$

where

$$e_{26} = \left(\frac{\rho}{\rho V_{2T} - M_2^l}\right), \qquad e_{53} = \left(\frac{\rho}{\rho V_{1T} - M_1^l}\right)$$

$$e_{66} = -\left(\frac{\rho}{\rho V_{2T} - M_2^l}\right), \qquad e_{76} = -\left(\frac{\rho(1-y_{A2})}{\rho V_{2T} - M_2^l}\right),$$

$$e_{86} = \left(\frac{\rho}{\rho V_{2T} - M_2^l}\right)$$

to obtain the following algebraic equations that can be solved for P_1 and P_2:

$$\begin{aligned} 0 &= P_1 - P^* \\ 0 &= P_2 - \frac{\rho M_2^v R T_2}{\rho V_{2T} - M_2^l} \end{aligned} \tag{63}$$

and six underlying constraints among the differential variables x:

$$\begin{aligned} 0 &= k_1^1(x) = P^* y_{A1} - P_{A1}^s x_{A1} \\ 0 &= k_2^1(x) = P^* - P_{A1}^s x_{A1} - P_{C1}^s(1 - x_{A1} - x_{B1}) \\ 0 &= k_3^1(x) = P^* - \frac{\rho M_1^v R T_1}{\rho V_{1T} - M_1^l} \\ 0 &= k_4^1(x) = \frac{\rho M_2^v R T_2}{\rho V_{2T} - M_2^l} - P_{A2}^s x_{A2} - P_{C2}^s(1 - x_{A2}) \\ 0 &= k_5^1(x) = \frac{\rho M_2^v R T_2(1 - y_{A2})}{\rho V_{2T} - M_2^l} - P_{C2}^s(1 - x_{A2}) \\ 0 &= k_6^1(x) = P^* - \frac{\rho M_2^v R T_2}{\rho V_{2T} - M_2^l} - \frac{1}{0.09}(F_{1v})^{7/4} \end{aligned} \tag{64}$$

Step 2. Differentiate the six constraints in x to obtain the new set of algebraic equations with the following form:

$$0 = \begin{bmatrix} \bar{k}_1^1 \\ \bar{k}_2^1 \\ \bar{k}_1^2 \\ \bar{k}_2^2 \\ \bar{k}_3^2 \\ \bar{k}_4^2 \\ \bar{k}_5^2 \\ \bar{k}_6^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \bar{l}_{11}^2 & \bar{l}_{12}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{l}_{21}^2 & \bar{l}_{22}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{l}_{31}^2 & \bar{l}_{32}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{l}_{44}^2 & \bar{l}_{45}^2 & l_{46}^2 & 0 & 0 \\ 0 & 0 & 0 & \bar{l}_{54}^2 & \bar{l}_{55}^2 & \bar{l}_{56}^2 & 0 & 0 \\ 0 & 0 & \bar{l}_{63}^2 & \bar{l}_{64}^2 & \bar{l}_{65}^2 & \bar{l}_{66}^2 & 0 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} N_{A1} \\ N_{C1} \\ \bar{F}_{1v} \\ N_{A2} \\ N_{C2} \\ F_{2v} \\ P_1 \\ P_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \bar{c}_{11}^2 & 0 & \bar{c}_{13}^2 \\ \bar{c}_{21}^2 & 0 & \bar{c}_{23}^2 \\ \bar{c}_{31}^2 & 0 & \bar{c}_{33}^2 \\ 0 & \bar{c}_{42}^2 & \bar{c}_{43}^2 \\ 0 & \bar{c}_{52}^2 & \bar{c}_{53}^2 \\ 0 & \bar{c}_{62}^2 & \bar{c}_{63}^2 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ F_{2l} \end{bmatrix}, \tag{65}$$

where the nonzero terms are functions of the differential variables x, the specific forms of which are omitted for brevity.

The iterative procedure converges in one iteration with $p_2 = p-1 = 7$ and $m_2 = p = 8$, and the algebraic equations in Eq. 65. Note that the third, fourth, and fifth algebraic equations in Eq. 65 involve only two algebraic variables N_{A1} and N_{C1}, implying the rank deficiency of the coefficient matrix for z, that is, $p_2 < p$. Premultiply these algebraic equations with a nonsingular matrix $E^2(x)$ of the form:

$$E^2(x) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & e_{83} & e_{84} & 1 & 0 & 0 & 0 \end{bmatrix},$$

where e_{83}, e_{84} are chosen such that

$$[e_{83} \quad e_{84}] \begin{bmatrix} \tilde{l}_{11}^2 & \tilde{l}_{12}^2 \\ \tilde{l}_{21}^2 & \tilde{l}_{22}^2 \end{bmatrix} = -\begin{bmatrix} \tilde{l}_{31}^2 & \tilde{l}_{32}^2 \end{bmatrix}$$

to obtain the final set of algebraic equations with the following form:

$$0 = \begin{bmatrix} \bar{k}_1^1 \\ \bar{k}_2^1 \\ \tilde{k}_1^2 \\ \tilde{k}_2^2 \\ \tilde{k}_4^2 \\ \tilde{k}_5^2 \\ \tilde{k}_6^2 \\ \hat{k}_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \tilde{l}_{11}^2 & \tilde{l}_{12}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \tilde{l}_{21}^2 & \tilde{l}_{22}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{l}_{44}^2 & \tilde{l}_{45}^2 & \tilde{l}_{46}^2 & 0 & 0 \\ 0 & 0 & 0 & \tilde{l}_{54}^2 & \tilde{l}_{55}^2 & \tilde{l}_{56}^2 & 0 & 0 \\ 0 & 0 & \tilde{l}_{63}^2 & \tilde{l}_{64}^2 & \tilde{l}_{65}^2 & \tilde{l}_{66}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} N_{A1} \\ N_{C1} \\ \bar{F}_{1v} \\ N_{A2} \\ N_{C2} \\ F_{2v} \\ P_1 \\ P_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \tilde{c}_{11}^2 & 0 & \tilde{c}_{13}^2 \\ \tilde{c}_{21}^2 & 0 & \tilde{c}_{23}^2 \\ 0 & \tilde{c}_{42}^2 & \tilde{c}_{43}^2 \\ 0 & \tilde{c}_{52}^2 & \tilde{c}_{53}^2 \\ 0 & \tilde{c}_{62}^2 & \tilde{c}_{63}^2 \\ \hat{c}_{11} & 0 & \hat{c}_{13} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ F_{2l} \end{bmatrix}, \quad (66)$$

where the last equation denotes a constraint in the differential variables x that involves the manipulated inputs $u_1 = Q_1$ and $u_3 = F_{2l}$, that is,

$$0 = \hat{k}_1(x) + \hat{c}_{11}(x)u_1 + \hat{c}_{13}(x)u_3. \quad (67)$$

Thus, the algorithmic procedure yields an equivalent DAE system comprised of the differential equations Eqs. 34–44 and 62 and the algebraic equations in Eq. 66, which is clearly not regular. In the next subsection, the proposed dynamic state-feedback regularizing compensator will be derived for this DAE system.

Dynamic feedback regularization and state-space realization

Consider the algebraic equations (Eq. 66) for the DAE system obtained from the algorithmic procedure. For this system, a choice of

$$M(x) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -r \end{bmatrix}, \quad (68)$$

with $r(x) = \dfrac{\hat{c}_{11}(x)}{\hat{c}_{13}(x)}$, and

$$S = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0] \quad (69)$$

satisfy the rank conditions desired in Eq. 15 and Eq. 21, respectively. Given these matrices, the proposed dynamic feedback regularizing compensator of Theorem 1 takes the following form:

$$\begin{aligned} \dot{w} &= v_1 \\ Q_1 &= v_3 \\ Q_2 &= v_2 \\ F_{2l} &= (\hat{c}_{13})^{-1}\left(-\hat{k}_1 + M_2^l + M_2^v + w - \hat{c}_{11}v_3\right), \end{aligned} \quad (70)$$

where $v = [v_1 \quad v_2 \quad v_3]^T$ is the new input vector and w is the state of the compensator.

Consider the DAE system with the differential equations in Eqs. 34–44 and 62 and the algebraic equations in Eq. 66, under the dynamic state-feedback compensator of Eq. 70. For the resulting DAE system with the extended vector of differential variables $\bar{x} = [x^T \quad w]^T$ and the new inputs v, the constraint in Eq. 67 becomes

$$0 = M_2^l + M_2^v + w.$$

Differentiating this constraint, the following set of algebraic equations is obtained:

$$0 = \begin{bmatrix} \tilde{k}_1 \\ \tilde{k}_2 \\ \tilde{k}_3 \\ \tilde{k}_4 \\ \tilde{k}_5 \\ \tilde{k}_6 \\ \tilde{k}_7 \\ \tilde{k}_8 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \tilde{l}_{11}^2 & \tilde{l}_{12}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \tilde{l}_{21}^2 & \tilde{l}_{22}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{l}_{44}^2 & \tilde{l}_{45}^2 & \tilde{l}_{46}^2 & 0 & 0 \\ 0 & 0 & 0 & \tilde{l}_{54}^2 & \tilde{l}_{55}^2 & \tilde{l}_{56}^2 & 0 & 0 \\ 0 & 0 & \tilde{l}_{63}^2 & \tilde{l}_{64}^2 & \tilde{l}_{65}^2 & \tilde{l}_{66}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} N_{A1} \\ N_{C1} \\ \bar{F}_{1v} \\ N_{A2} \\ N_{C2} \\ F_{2v} \\ P_1 \\ P_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tilde{c}_{11}^2 - r\tilde{c}_{13}^2 \\ 0 & 0 & \tilde{c}_{21}^2 - r\tilde{c}_{23}^2 \\ 0 & \tilde{c}_{42}^2 & -r\tilde{c}_{43}^2 \\ 0 & \tilde{c}_{52}^2 & -r\tilde{c}_{53}^2 \\ 0 & \tilde{c}_{62}^2 & -r\tilde{c}_{63}^2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad (71)$$

where the coefficient matrix for the algebraic variables z has full rank p. Thus, these algebraic equations can be solved for the algebraic variables z in terms of the differential variables $\bar{x}$ and the inputs v. Note that the solution for N_{A1}, N_{C1} involve only the input v_3, while the solution for N_{A2}, N_{C2}, F_{2V}, $\bar{F}_{1v}$ involve all three inputs v_1, v_2, v_3.

A substitution of the resulting solution for the algebraic variables into the differential equations (Eqs. 34–44 and 62) yields the state-space realization of Proposition 1. More specifically, in view of the nature of the solutions for the algebraic variables mentioned earlier, the state-space realization has the following form:

$$\begin{aligned}
\dot{\bar{x}} &= \bar{f}_1(\bar{x}) + \bar{g}_{1,3}(\bar{x}) v_3 \\
&\vdots \\
\dot{\bar{x}}_6 &= \bar{f}_6(\bar{x}) + \bar{g}_{6,3}(\bar{x}) v_3 \\
\dot{\bar{x}}_7 &= \bar{f}_7(\bar{x}) + \bar{g}_{7,1}(\bar{x}) v_1 + \bar{g}_{7,2}(\bar{x}) v_2 + \bar{g}_{7,3}(\bar{x}) v_3 \\
&\vdots \\
\dot{\bar{x}}_{12} &= \bar{f}_{12}(\bar{x}) + \bar{g}_{12,1}(\bar{x}) v_1 + \bar{g}_{12,2}(\bar{x}) v_2 + \bar{g}_{12,3}(\bar{x}) v_3 \\
\dot{\bar{x}}_{13} &= v_1 \\
y_1 &= \bar{x}_6 \\
y_2 &= \bar{x}_{10} \\
y_3 &= \bar{x}_7
\end{aligned} \quad (72)$$

where $\bar{x}_{13} = w$, and

$$\bar{x} \in \mathfrak{M} = \left\{ (x,w) \in \mathfrak{X} \times \mathbb{R}: \begin{array}{r} k_1^1(x) = 0 \\ \vdots \\ k_6^1(x) = 0 \\ x_7 + x_9 + w = 0 \end{array} \right\}.$$

This state-space realization will be used as the basis for the synthesis of the proposed feedback controller.

Feedback controller synthesis and control performance

Consider the state-space realization in Eq. 72 of the feedback regularized DAE system. It can be easily verified that the relative orders of the controlled outputs y_1, y_2, y_3 with respect to the inputs v_1, v_2, v_3 are

$$r_1 = 1, \qquad r_2 = 1, \qquad r_3 = 1,$$

and the corresponding characteristic matrix is nonsingular with the following form:

$$C(\bar{x}) = \begin{bmatrix} 0 & 0 & \bar{g}_{6,3}(\bar{x}) \\ \bar{g}_{10,1}(\bar{x}) & \bar{g}_{10,2}(\bar{x}) & \bar{g}_{10,3}(\bar{x}) \\ \bar{g}_{7,1}(\bar{x}) & \bar{g}_{7,2}(\bar{x}) & \bar{g}_{7,3}(\bar{x}) \end{bmatrix}.$$

Thus, an input/output decoupled response of the following form was requested:

$$\beta_{i0}^i y_i + \beta_{i1}^i \frac{dy_i}{dt} = \tilde{v}_i, \qquad i = 1,2,3 \quad (73)$$

and the corresponding dynamic state feedback controller of Theorem 2 is given by

$$\begin{aligned}
\dot{w} &= v_1 \\
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \left\{ \begin{bmatrix} \beta_{11}^1 & 0 & 0 \\ 0 & \beta_{21}^2 & 0 \\ 0 & 0 & \beta_{31}^3 \end{bmatrix} C(\bar{x}) \right\}^{-1} \begin{bmatrix} \tilde{v}_1 - \beta_{10}^1 x_6 - \beta_{11}^1 \bar{f}_6(\bar{x}) \\ \tilde{v}_2 - \beta_{20}^2 x_{10} - \beta_{21}^2 \bar{f}_{10}(\bar{x}) \\ \tilde{v}_3 - \beta_{30}^3 x_7 - \beta_{31}^3 \bar{f}_7(\bar{x}) \end{bmatrix} \\
Q_1 &= v_3 \\
Q_2 &= v_2 \\
F_{2l} &= (\hat{c}_{13})^{-1}\left(-\hat{k}_1 + M_2^l + M_2^v + w - \hat{c}_{11} v_3 \right).
\end{aligned} \quad (74)$$

Finally, an error feedback controller with integral action, with the following realization (Daoutidis and Kravaris, 1994):

$$\left. \begin{aligned} \dot{\xi}_i &= \frac{(y_{spi} - y_i)}{\gamma_{i1}^i} \\ \tilde{v}_i &= \beta_{i0}^i \xi_i + \frac{\beta_{i1}^i}{\gamma_{i1}^i}(y_{spi} - y_i) \end{aligned} \right\} \qquad i = 1,2,3$$

was implemented around the linear $\tilde{v} - y$ system of Eq. 73 to induce the overall closed-loop input/output behavior:

$$y_i + \gamma_{i1}^i \frac{dy_i}{dt} = y_{spi}, \qquad i = 1,2,3. \quad (75)$$

The controller was tuned with the following parameters:

$$\begin{gathered}
\beta_{10}^1 = 10, \qquad \beta_{11}^1 = 900 \text{ s} \\
\beta_{20}^2 = 10, \qquad \beta_{21}^2 = 600 \text{ s} \\
\beta_{30}^3 = 10, \qquad \beta_{31}^3 = 1{,}200 \text{ s} \\
\gamma_{11}^1 = 900 \text{ s} \qquad \gamma_{21}^2 = 600 \text{ s} \qquad \gamma_{31}^3 = 1{,}200 \text{ s},
\end{gathered}$$

and its performance was compared through simulations with that of a controller designed on the basis of the index-two

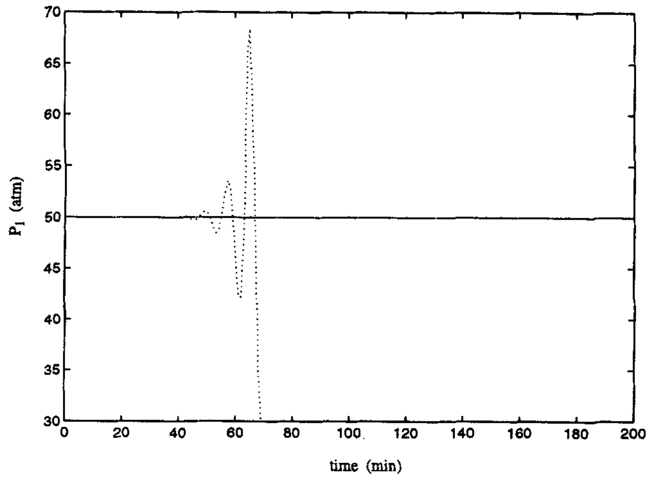

Figure 6. Closed-loop profiles of reactor pressure P_1, with (solid) and without (dotted) the proportional feedback controller.

DAE model that retains the fast pressure dynamics of the reactor. In both cases, the detailed index-one model was used to simulate the process.

The first simulation run shows that the reactor pressure P_1 has an unstable dynamics. Starting at the nominal steady state, the closed-loop system with the preceding controller was simulated in the absence of the proportional controller of Eq. 56, that is, with the controller gain K set to zero. Clearly, the corresponding profile in Figure 6 shows that the reactor pressure is unstable and needs to be stabilized. The next run shows that under the proportional feedback controller of Eq. 56, the fast dynamics of the reactor pressure P_1 is stable. Again, starting from the nominal steady state, a 10% increase in the desired value P^* was imposed at $t = 10$ min, and the corresponding closed-loop profiles are shown in Figure 7. Clearly, the reactor pressure P_1 rises quickly to the new desired level. Moreover, the profile for the product flow rate F_{2v} calculated from the controller relation (Eq. 56) is compared with that evaluated from the pseudo-steady-state constraint $P_1 = P^*$, obtained by ignoring the pressure dynamics. The two profiles are indeed close, justifying the assumption of fast and negligible pressure dynamics. Similar comparisons for the interphase mole transfer rates N_{A1}, N_{C1}, N_{A2}, N_{C2} evaluated from the explicit mass-transfer correlations (Eqs. 45–48) and the phase-equilibrium relations (Eqs. 57–60) are also shown. The profiles clearly illustrate the validity of the phase-equilibrium conditions obtained by ignoring the fast mass transfer.

The third run studied the setpoint tracking capabilities of the proposed controller in the nominal case, that is, without any modeling errors or disturbances. For the process that was initially at its nominal steady state, a 10% decrease in y_{sp2}, the setpoint for desired product composition y_{A2}, and a 1% decrease in y_{sp1}, the setpoint for the reactor temperature T_2 was imposed at $t = 10$ min. The closed-loop profiles of the three controlled outputs and manipulated inputs are shown in Figure 8. The performance of the proposed controller is virtually indistinguishable from that of the index-two model-based controller. Thus, the proposed index-three model-based controller, which ignores the inherent fast and stable modes of the process, yields excellent performance in the nominal case.

The fourth run compared the performance of the two controllers for the same setpoint changes, in the presence of small parametric uncertainties. More specifically, the preceding setpoint changes were imposed with a 5% error in the values for the molar heat capacity c_p and the liquid molar density ρ. As shown in Figure 9, the index-three model-based controller asymptotically rejects the effects of these errors, with a slight performance degradation as expected. Moreover, the manipulated input profiles in the presence of these modeling errors are close to the corresponding profiles in the nominal case (see Figure 8). On the other hand, the controller based on the index-two model is highly ill-conditioned and the calculated control action showed large initial overshoots, before returning to the final steady-state values. The corresponding output profiles also show a significant deterioration in the controller performance. It should be observed that this controller ill-conditioning will get more pronounced, that is, the calculated control action will become more sensitive to small errors, in the presence of faster pressure dynamics. In contrast, the proposed index-three model-based controller that ignores the fast modes, is well-conditioned and does not exhibit a high sensitivity to small modeling errors.

The last run addressed the ability of the proposed controller to reject unmeasured disturbances in the inlet flow rates of the reactants A and B. Again, for the process at its nominal steady state, a 10% decrease in y_{sp1} and a 1% decrease in y_{sp2} were imposed at $t = 10$ min , while a 5% disturbance in the inlet flow rates F_A, F_B was imposed at $t = 20$ min. As illustrated in the closed-loop profiles in Figure 10, the controller successfully rejects the effects of these unmeasured disturbances with very good performance. The controller based on the index-two model also is well-conditioned with respect to these disturbances, since the effects of small errors in F_A and F_B are not magnified in the control action, owing to the specific structure of the process.

Conclusions

This article addressed the feedback control of nonlinear high-index DAE systems of the form in Eq. 1, for which the underlying constraints in the differential variables involve the manipulated inputs. In contrast with DAE systems of the form of Eq. 1 for which the underlying constraints in the differential variables are independent of the inputs (Kumar and Daoutidis, 1995a), a state-space realization of such systems cannot be derived independently of the control law for the manipulated inputs. Motivated by this, a control methodology was developed, which involved, as a key step, a feedback modification of the DAE system such that the resulting system is regular, that is, possesses a state-space realization independently of the control law for the new inputs. More specifically, an algorithmic procedure was developed initially to identify the underlying constraints and yield an equivalent system where the algebraic equations explicitly include the constraints that involve the manipulated inputs in a nonsingular fashion, thereby isolating the cause of nonregularity. The resulting system was then modified through a dynamic feedback compensator to obtain an extended system that is regular. Finally, a state-space realization of the feedback reg-

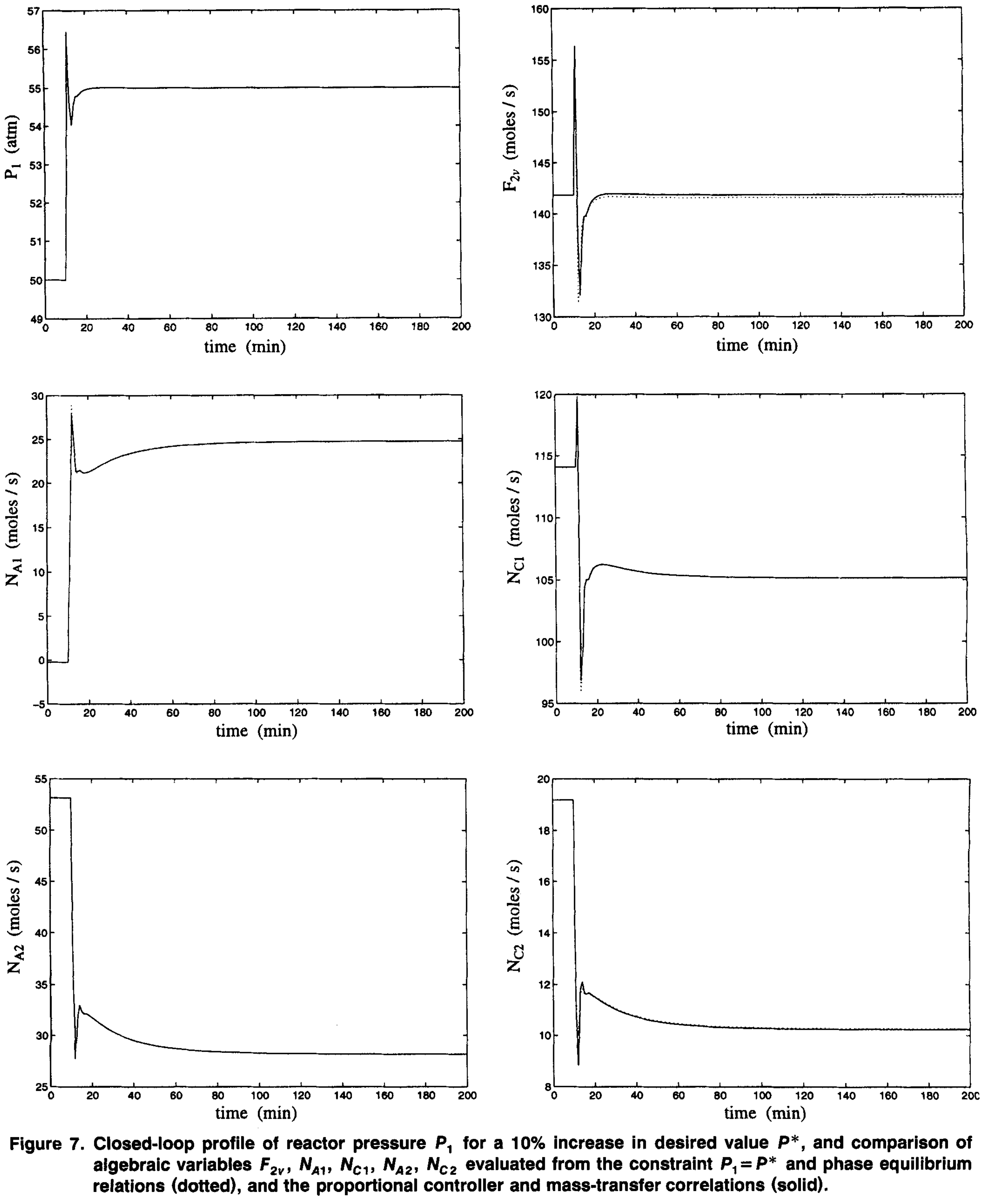

Figure 7. Closed-loop profile of reactor pressure P_1 for a 10% increase in desired value P^*, and comparison of algebraic variables F_{2v}, N_{A1}, N_{C1}, N_{A2}, N_{C2} evaluated from the constraint $P_1 = P^*$ and phase equilibrium relations (dotted), and the proportional controller and mass-transfer correlations (solid).

ularized system was derived and used as the basis for the synthesis of a dynamic state-feedback controller that induces a well-characterized, linear input/output behavior in the closed-loop system. The proposed control methodology for nonregular DAE systems, together with the control methodology developed in Kumar and Daoutidis (1995a) for regular

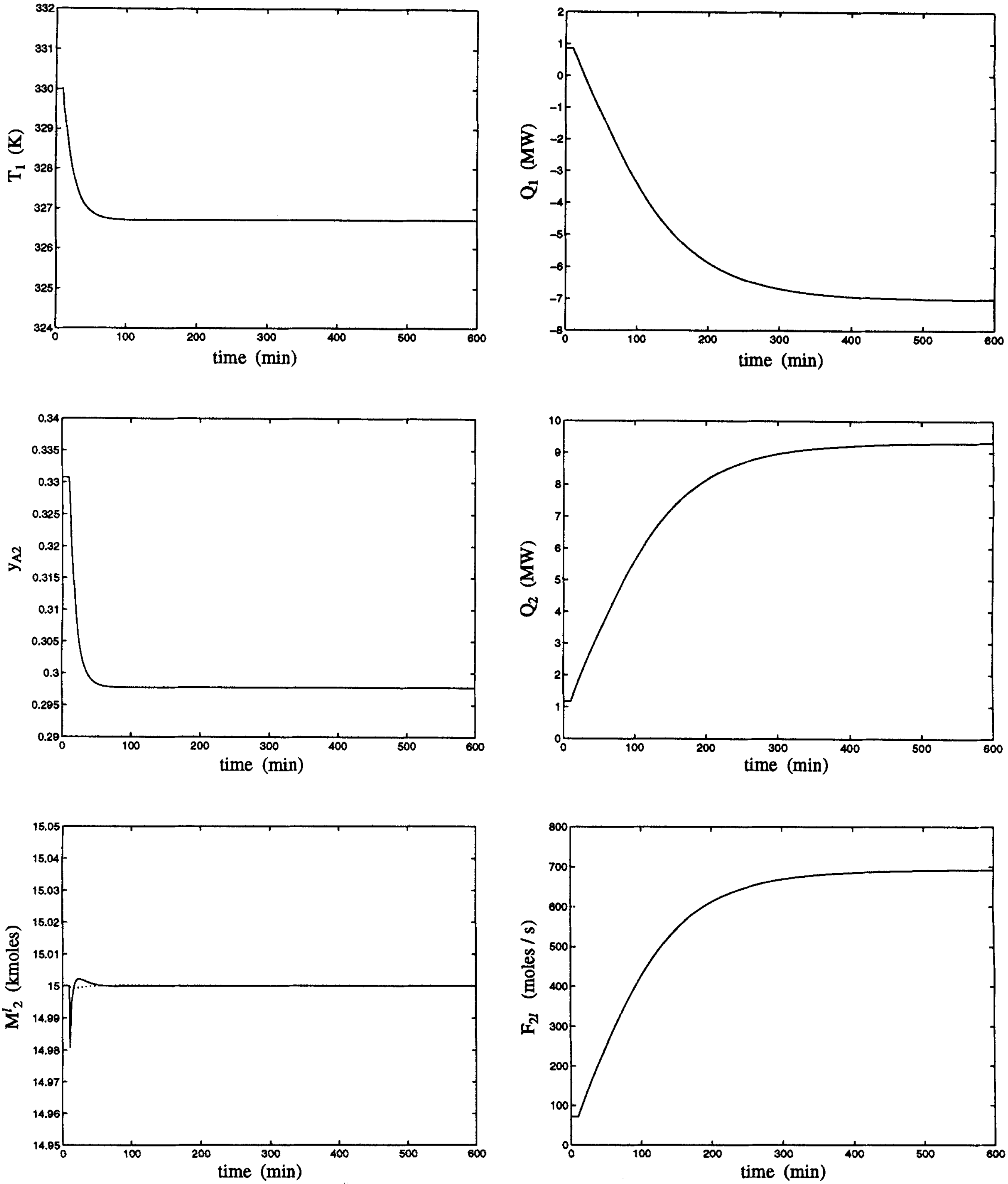

Figure 8. Closed-loop profiles of controlled outputs and manipulated inputs for a 1% decrease in y_{sp1} and 10% decrease in y_{sp2}, under index-3 (solid) and index-2 (dotted) DAE model-based controllers.

DAE systems, comprise a comprehensive methodological framework for the feedback control of nonlinear high-index DAE systems.

It was shown that high-index DAE systems arise naturally as dynamic models of a wide variety of chemical processes, by ignoring the inherent fast and stable modes. In this work, an

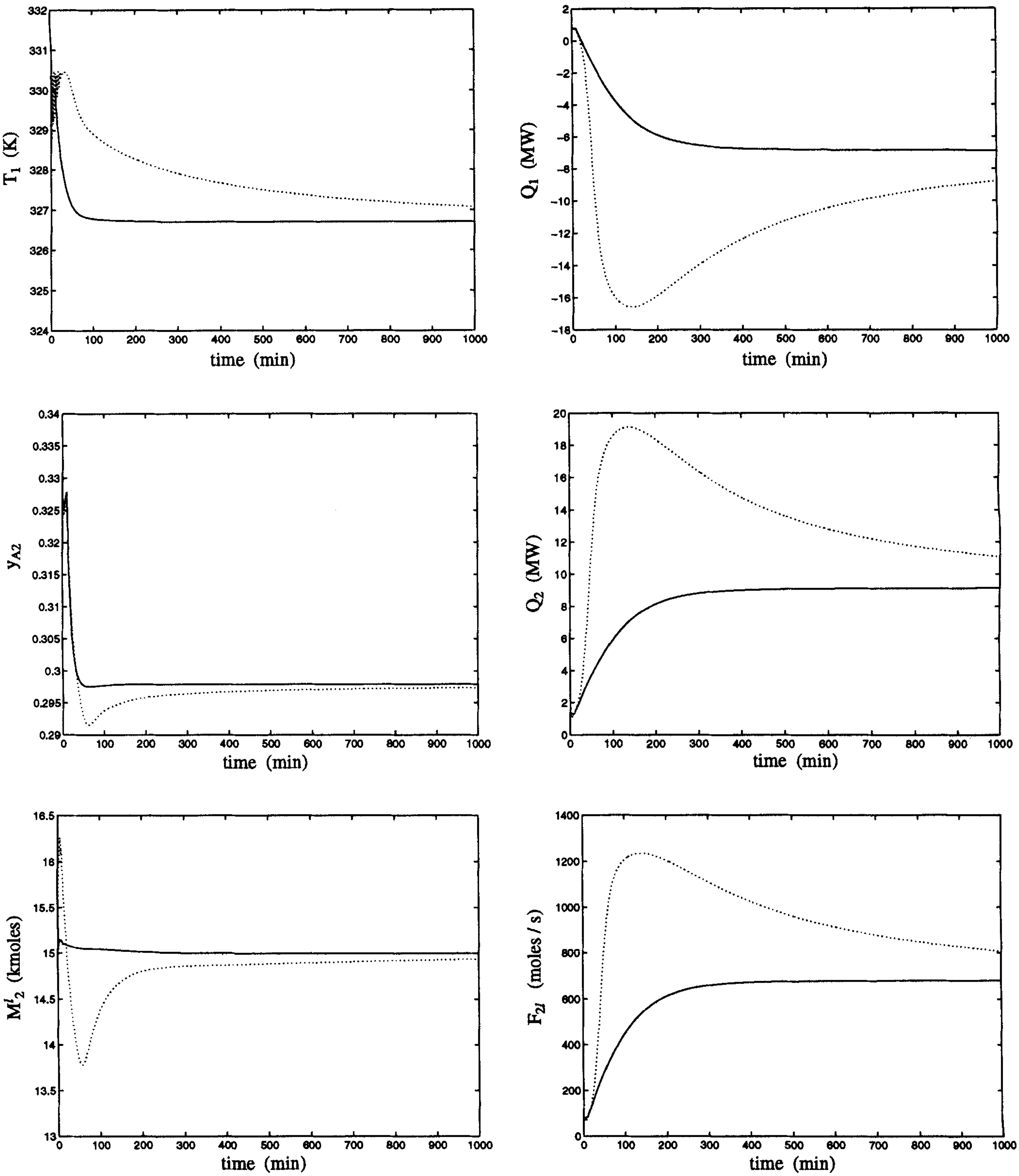

Figure 9. Closed-loop profiles of controlled outputs and manipulated inputs for a 1% decrease in y_{sp1} and 10% decrease in y_{sp2} in the presence of parametric uncertainties, under index-3 (solid) and index-2 (dotted) DAE model-based controllers.

interconnection of a two-phase reactor and a condenser was considered, with a fast and stable interphase mass transfer and a fast reactor pressure dynamics that is unstable. The fast pressure dynamics was stabilized through a proportional feedback controller. A detailed model of the process with explicit correlations for the fast mass transfer, and the propor-

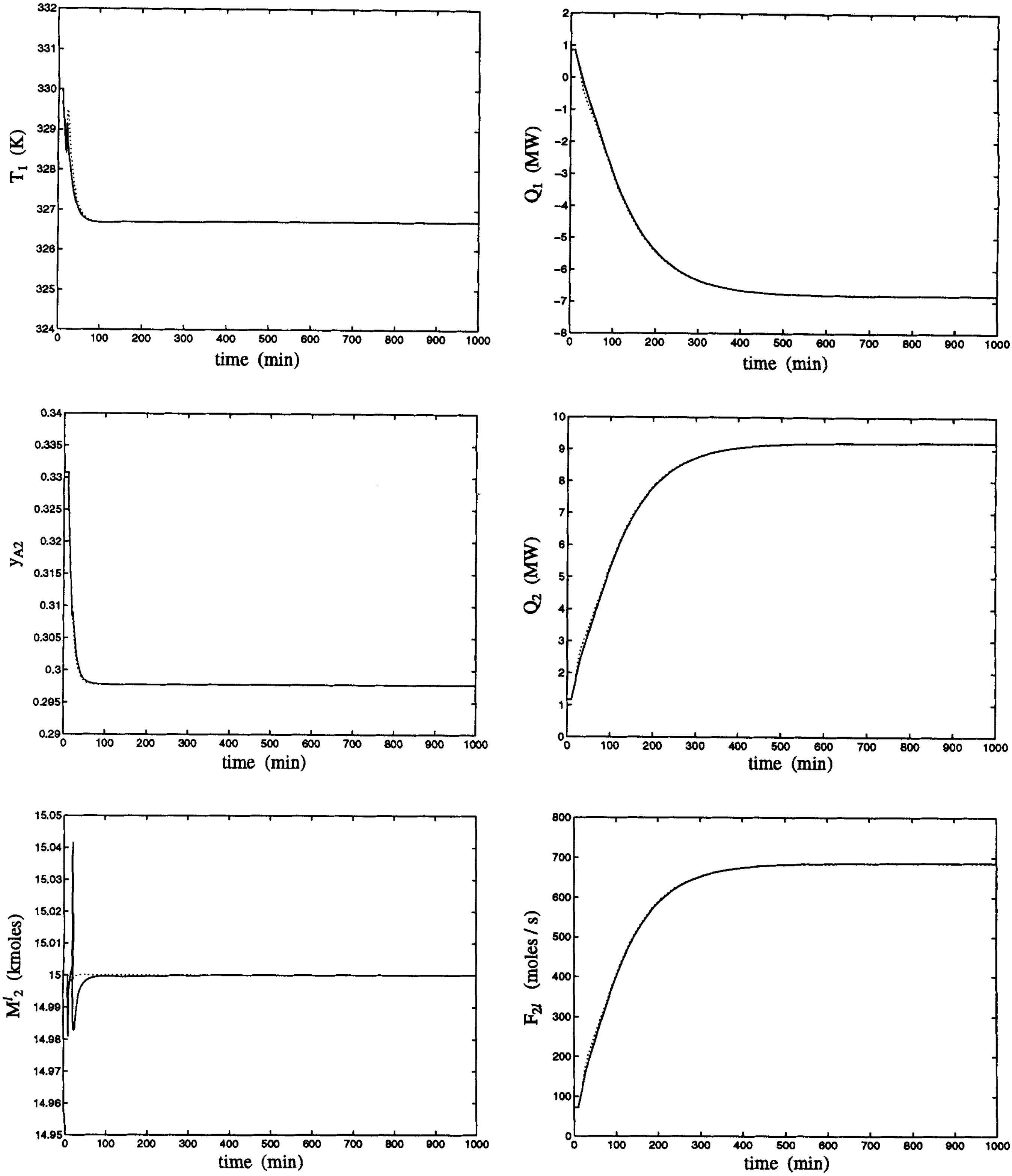

Figure 10. Closed-loop profiles of controlled outputs and manipulated inputs for a 1% decrease in y_{sp1} and 10% decrease in y_{sp2} in the presence of disturbances, under index-3 (solid) and index-2 (dotted) DAE model-based controllers.

tional feedback controller for the reactor pressure, is a DAE system of index-one. Controllers designed on the basis of this detailed model ignore the inherent time-scale-multiplicities in the process and suffer from ill-conditioning and closed-loop instability. On the other hand, an approximate model that describes only the slow dynamics of the process, that is, ignores the fast and stable mass transfer and reactor pressure dynamics, is an index-three DAE system that is nonregular. The controller derived on the basis of this model shows excellent performance and is well-conditioned, that is, it does not exhibit a high sensitivity to small errors.

Acknowledgment

Financial support for this work from the National Science Foundation, grant CTS-9320402, is gratefully acknowledged.

Notation

$\mathbb{R}$ = real line

$\mathbb{R}^i$ = i-dimensional Euclidean space

$[\cdot]^T$ = transpose of a vector/matrix

$L_f\alpha(x)$ = Lie derivative of a scalar field $\alpha(x)$ ($x \in \mathbb{R}^n$) with respect to an n-dimensional vector field $f(x)$, defined as $L_f\alpha(x) = [(\partial\alpha/\partial x_1)\cdots(\partial\alpha/\partial x_n)]f(x)$

$L_f^j\alpha(x)$ = high-order Lie derivative defined as $L_f^j\alpha(x) = L_f(L_f^{j-1}\alpha(x))$, $j = 2, 3, \ldots$

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Appendix

Proof of Lemma 1

Consider a solvable DAE system of the form of Eq. 18, with a finite index $\nu_d > 1$ for a smooth input $u(t)$. For simplicity, consider $u(t) = \alpha$, where $\alpha \in \mathbb{R}^m$ is a constant vector. The corresponding DAE system is described by

$$\begin{aligned} \dot{x} &= \bar{f}(x) + b(x)z \\ 0 &= \bar{k}(x) + l(x)z, \end{aligned} \tag{A1}$$

where $\bar{f}(x) = f(x) + g(x)\alpha$, $\bar{k}(x) = k(x) + c(x)\alpha$, and rank $l(x) = p_1 < p$. Then, by the definition of index, a set of differential equations for z in the DAE system of Eq. 76 can be obtained by differentiating a proper subset of the algebraic equations ν_d times. In particular, the following algorithmic procedure will converge in exactly $s = \nu_d - 1$ iterations to yield a set of algebraic equations that are nonsingular with respect to the algebraic variables z (see Kumar and Daoutidis, 1995a).

Iteration 1

Step 1. Premultiply the algebraic equations in Eq. A1 by a $p \times p$ nonsingular matrix $E^1(x)$ such that:

$$E^1(x)l(x) = \begin{bmatrix} \bar{l}^1(x) \\ 0 \end{bmatrix},$$

where the $p_1 \times p$ matrix $\bar{l}^1(x)$ has full row rank, to obtain:

$$0 = \begin{bmatrix} \bar{k}^1(x) \\ \hat{k}^1(x) \end{bmatrix} + \begin{bmatrix} \bar{l}^1(x) \\ 0 \end{bmatrix} z,$$

where $\bar{k}^1(x)$, $\hat{k}^1(x)$ are vector fields of dimensions p_1 and $(p-p_1)$, respectively.

Step 2. Differentiate the last $p-p_1$ algebraic equations, that is, the constraints $\hat{k}^1(x)=0$ among x, to obtain the following algebraic equations:

$$0=\begin{bmatrix}\bar{k}^1(x)\\ \tilde{k}^2(x)\end{bmatrix}+\begin{bmatrix}\bar{l}^1(x)\\ \tilde{l}^2(x)\end{bmatrix}z \tag{A2}$$

where

$$\tilde{k}^2(x)=\left[\frac{d\hat{k}^1(x)}{dx}\right]\tilde{f}(x),\qquad \tilde{l}^2(x)=\left[\frac{d\hat{k}^1(x)}{dx}\right]b(x).$$

Step 3. Evaluate the rank p_2 of the matrix:

$$\begin{bmatrix}\bar{l}^1(x)\\ \tilde{l}^2(x)\end{bmatrix}$$

where $p_2 \geq p_1$. If $p_2=p$, then stop, else proceed to next iteration.

Iteration q $(q \geq 2)$

Step 1. Consider the algebraic equation from the iteration $q-1$:

$$0=\begin{bmatrix}\bar{k}^{q-1}(x)\\ \tilde{k}^q x\end{bmatrix}+\begin{bmatrix}\bar{l}^{q-1}(x)\\ \tilde{l}^q(x)\end{bmatrix}z, \tag{A3}$$

where

$$\operatorname{rank}\begin{bmatrix}\bar{l}^{q-1}(x)\\ \tilde{l}^q(x)\end{bmatrix}=p_q<p.$$

Premultiply these equations with a nonsingular $p \times p$ matrix $E^q(x)$ such that:

$$E^q(x)\begin{bmatrix}\bar{l}^{q-1}(x)\\ \tilde{l}^q(x)\end{bmatrix}=\begin{bmatrix}\bar{l}^q(x)\\ 0\end{bmatrix},$$

where the $p_q \times p$ matrix $\bar{l}^q(x)$ has full row rank, to obtain:

$$0=\begin{bmatrix}\bar{k}^q(x)\\ \hat{k}^q(x)\end{bmatrix}+\begin{bmatrix}\bar{l}^q(x)\\ 0\end{bmatrix}z, \tag{A4}$$

where $\bar{k}^q(x)$, $\hat{k}^q(x)$ are vector fields of dimensions p_q, $(p-p_q)$, respectively.

Step 2. Differentiate the last $p-p_q$ equations to obtain:

$$0=\begin{bmatrix}\bar{k}^q(x)\\ \tilde{k}^{q+1}(x)\end{bmatrix}+\begin{bmatrix}\bar{l}^q(x)\\ \tilde{l}^{q+1}(x)\end{bmatrix}z, \tag{A5}$$

where

$$\tilde{k}^{q+1}(x)=\left[\frac{d\hat{k}^q(x)}{dx}\right]\tilde{f}(x),\qquad \tilde{l}^{q+1}(x)=\left[\frac{d\hat{k}^q(x)}{dx}\right]b(x).$$

Step 3. Evaluate the rank p_{q+1} of the matrix

$$\begin{bmatrix}\bar{l}^q(x)\\ \tilde{l}^{q+1}(x)\end{bmatrix},$$

where $p_{q+1} \geq p_q$. If $p_{q+1}=p$, then stop, else proceed to the next iteration.

For a DAE system in Eq. A1 with a finite index ν_d, the algorithmic procedure will converge in exactly $s=\nu_d-1$ iterations with $p_{s+1}=p$. With the aid of this result, we now prove that for the preceding algorithmic procedure to converge, it is necessary that the matrix

$$\begin{bmatrix}l(x)\\ b(x)\end{bmatrix}$$

has full column rank p.

Assume that

$$\operatorname{rank}\begin{bmatrix}l(x)\\ b(x)\end{bmatrix}=p^*<p. \tag{A6}$$

Then, through induction it will be shown that $p_i \leq p^* < p$ for every iteration i of the previous algorithm, hence proving the assumption to be wrong.

The assumption in Eq. A6 leads to the following two conclusions:

(a) For the choice of matrix $E^1(x)$ as in the preceding algorithm:

$$\operatorname{rank}\begin{bmatrix}b(x)\\ \bar{l}^1(x)\\ 0\end{bmatrix}=\operatorname{rank}\left\{\begin{bmatrix}I_n & 0\\ 0 & E^1(x)\end{bmatrix}\begin{bmatrix}b(x)\\ l(x)\end{bmatrix}\right\}\leq p^*, \tag{A7}$$

or

$$\operatorname{rank}\begin{bmatrix}\bar{l}^1(x)\\ b(x)\end{bmatrix}\leq p^*<p.$$

(b) If, for any $i \geq 1$,

$$\operatorname{rank}\begin{bmatrix}\bar{l}^i(x)\\ b(x)\end{bmatrix}\leq p^*<p,$$

then

(1) according to the algorithmic procedure previously outlined

$$\operatorname{rank}\begin{bmatrix}\bar{l}^i(x)\\ \tilde{l}^{i+1}(x)\end{bmatrix}=\operatorname{rank}\left\{\begin{bmatrix}I_{p_i} & 0\\ 0 & \dfrac{d\hat{k}^i(x)}{dx}\end{bmatrix}\begin{bmatrix}\bar{l}^i(x)\\ b(x)\end{bmatrix}\right\}\leq p^*$$

or

$$\operatorname{rank}\begin{bmatrix} \bar{l}^i(x) \\ \bar{l}^{i+1}(x) \end{bmatrix} = p_{i+1} \le p^* < p;$$

(2) consider the matrix:

$$E^{i+1}(x) = \begin{bmatrix} E_1^{i+1}(x) \\ E_2^{i+1}(x) \end{bmatrix}$$

where $E_1^{i+1}(x)$ is the matrix consisting of the first p_{i+1} rows of $E^{i+1}(x)$, that is,

$$E_1^{i+1}(x)\begin{bmatrix} \bar{l}^i(x) \\ \tilde{l}^{i+1}(x) \end{bmatrix} = \bar{l}^{i+1}(x).$$

Thus,

$$\operatorname{rank}\begin{bmatrix} \bar{l}^{i+1}(x) \\ \cdots\cdots \\ b(x) \end{bmatrix} = \operatorname{rank}\left\{ \begin{bmatrix} E^{i+1}(x) & 0 \\ \cdots\cdots & \cdots\cdots \\ 0 & I_n \end{bmatrix} \times \begin{bmatrix} I_{p_i} & 0 \\ 0 & \dfrac{d\hat{k}^i(x)}{dx} \\ \cdots\cdots & \cdots\cdots \\ 0 & I_n \end{bmatrix} \begin{bmatrix} \bar{l}^i(x) \\ b(x) \end{bmatrix} \right\} \le p^* < p.$$

Clearly, from conclusions (a) and (b), it follows that $p_i \le p^* < p$ for every iteration $i \ge 1$ in the preceding algorithm, thus, contradicting the assumption in Eq. A6.

Proof of Proposition 1

Consider the feedback regularized extended DAE system in Eq. 24. In Theorem 1, it was established that for consistent initial conditions $\bar{x}(0) \in \mathfrak{M}$, the solution $\bar{x}(t)$ of Eq. 24 is constrained to evolve in $\mathfrak{M}$ for all times, with the corresponding solution for z in Eq. 26. A direct substitution of the solution for z in the differential equations for $\bar{x}$ yields the differential equations in Eq. 28 on the constrained state space $\mathfrak{M}$, for which the solution $\bar{x}(t)$ is the same as that for the DAE system in Eq. 24.

Proof of Theorem 2

Consider the DAE system in Eq. 1 and the dynamic state-feedback control law of Eq. 32. The proposed algorithmic procedure yields the DAE system of Eq. 12, which is equivalent to the DAE system in Eq. 1 in the sense that both systems have the same solutions $x(t)$, $z(t)$, and hence the same outputs $y_i(t)$ for *any* (smooth) input $u(t)$. Thus, the closed-loop DAE system of Eq. 1 under the control law of Eq. 32 is equivalent (in the preceding sense) to the closed-loop DAE system of Eq. 12 under the same control law (Eq. 32), and it suffices to prove the result for the latter system.

First, consider the DAE system of Eq. 12 under the control law for the manipulated inputs u in terms of the new inputs $v = [v_1^T \;\; v_2^T]^T$ in Eq. 23, that is, the extended DAE system in Eq. 24. Differentiating the last $p - p_{s+1}$ algebraic equations in Eq. 24 once, we obtain the equivalent DAE system:

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} \bar{f}(x,w) \\ 0 \end{bmatrix} + \begin{bmatrix} b(x) \\ 0 \end{bmatrix} z + \begin{bmatrix} 0 & \bar{g}_2(x) \\ I_{p-p_{s+1}} & 0 \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ v_2 \end{bmatrix}$$

$$0 = \begin{bmatrix} \tilde{k}(x,w) \\ S\bar{f}(x,w) \end{bmatrix} + \begin{bmatrix} \bar{l}(x) \\ Sb(x) \end{bmatrix} z + \begin{bmatrix} 0 & \bar{c}_2(x) \\ I_{p-p_{s+1}} & S\bar{g}_2(x) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$y_i = h_i(x), \qquad i = 1, \ldots, m, \tag{A8}$$

$\bar{x} = [x^T \;\; w^T]^T \in \mathfrak{M}$ for any inputs v. Moreover, from Proposition 1, the preceding DAE system (Eq. A8) has an equivalent state-space realization of Eq. 28 for *any* input v, and in particular the specific control law given by

$$v = \{[\,\beta_{1r_1} \cdots \beta_{mr_m}]C(\bar{x})\}^{-1}\left(\bar{v} - \sum_{i=1}^{m}\sum_{j=0}^{r_i} \beta_{ij} L_{\bar{f}}^{j} \bar{h}_i(\bar{x})\right).$$

Thus, the DAE system of Eq. 1 under the overall dynamic state-feedback control law of Eq. 32 is equivalent to the ODE system described by

$$\dot{\bar{x}} = \bar{f}(\bar{x}) + \bar{g}(\bar{x})\{[\,\beta_{1r_1} \cdots \beta_{mr_m}]C(\bar{x})\}^{-1} \times \left(\bar{v} - \sum_{i=1}^{m}\sum_{j=0}^{r_i} \beta_{ij} L_{\bar{f}}^{j} \bar{h}_i(\bar{x})\right)$$

$$y_i = \bar{h}_i(\bar{x}), \qquad i = 1, \ldots, m \tag{A9}$$

on the constrained state-space $\mathfrak{M}$. For the preceding system, by obtaining the expressions for the derivatives of the outputs, i.e., $(d^j y_i/dt^j)$, $i = 1, \ldots, m$; $j = 1, \ldots, r_i$, it can be verified that the desired input/output response in Eq. 31 is indeed enforced in the closed-loop system.

Manuscript received Mar. 13, 1995, and revision received Dec. 18, 1995.